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Introduction to Algebra

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In this chapter, we will look at the fundamental laws of Algebra. These laws mainly define

- ➔ how to use the signs $+, -, \div$
- ➔ how to manipulate numbers (for instance, 1.3 or $\frac{1}{2}$) and symbols (for instance, a or x)

As always, please free to refer to the book [Croft and Davidson, 2016] for details.

I Introduction

During the ancient times, mathematicians used to solve applied problems. They were hence manipulating real quantities (say a circle of radius 1.5, or a square with a side 3). Quickly, though, they realized that they were using similar methods to find the solution of similar problems. For instance, the perimeter of a square of side 1 is 4×1, and, if the side is 2, the perimeter is then 4×2.

The need of using *symbols* hence risen. It allowed to solve a class of problems, without considering a particular problems. Noting L the side, the perimeter of a square is then 4L, and this is true whatever L is. L is known as a symbol, or a variable.

In mathematics unknown quantities are given symbols, usually x, y or z. Since Pascal, it is usual to use letters at the end of the alphabet to denote unknown quantities x, y etc. Letters at the beginning of the alphabet are used to denote constants.

These unknown quantities are then manipulated using the *laws of algebra*.

I a) Some notations

Note that

- ➔ $6x = 6 \times x$
- ➔ x really means $1x$ but we omit the 1
- ➔ $xy = x \times y$ and can just as easily be written as yx but it is usual to write them in alphabetical order.

It also helps if you write a curly x to differenti ate it from the multiplication sign $\times$.

II Laws of Algebra

II a) Addition and Subtraction

Only like terms can be added or subtracted.

Def.

LIKE TERMS: Like terms are terms that contain:

1. the same variables
2. this variable have to be raised to the same power

Main Example

That can be visualize on table. If it is after for breakfast, you might have 4 bowls and maybe 3 spoons. We cannot add the bowls to the spoons, they are not the same kind of things. If our sister go get in the kitchen a few others things, like another 3 bowls and 5 spoons. There is now have 7 bowls and 8 spoons.
Next, the father comes to clear up a bit and grabs 2 bowls. We are left with 5 bowls, but the number of spoons is unchanged.
Similarly with algebra, we can only add (or subtract) similar "objects", or those with the same letter raised to the same power.
Hence, if we note x as bowls and y as spoons:

$$\underbrace{4x+3y}_{\text{from breakfast}} \quad \underbrace{+3x+5y}_{\text{sister}}$$

then we can gather the x's together and gather the y's together to give

$$\underbrace{4x+3x}_{\text{bowls}}+\underbrace{3y+5y}_{\text{spoons}}=7x+8y$$

More illustrations of this first law of algebra:

1. $2x+2x+y=4x+y$
   we fist gather the xs: $2x+2x=(2+2)x=4x$ and then let the y alone
2. $2x+2x+y+2y=4x+3y$
   we fist gather the xs as before, then gather the ys: $y+2y=(1+2)y=3y$.
3. $2x+5y+3y+y+3x=5x+9y$
   here xs are gathered together, then ys and finally zs
4. $2x+5y+x-2y+4=3x+3y+4$
   note that the 4 remains aside
5. $7x-5y+x+2y=8x-3y$
   note that $-5+2=2-5=-3$
6. $x+5y-4x+5y=-3x+10y$
   this can also be written as $10y-3x$; it is common practice to start with a positive number, but $-3x+10y$ correct as well.

Exercise 1.
Simplify these expressions

1.1 $7x+4x-x$	1.2 $3x-2+x$
1.3 $5x+3x+6$	1.4 $x-3x+5x+12x$
1.5 $2y+5y+3y-y$	1.6 $y-3y+6+4y$
1.7 $2a+3b+4a-3b$	1.8 $3x-(-4x)+5x+2$
1.9 $5x-2x-(-4x)-7x$	1.10 $8x-3x+2y-(-x)+(-5y)$
1.11 $7x-9-3x+x-10x$	1.12 $x^2+3x^2-5y+y^2$
1.13 $3y-3x+4-y-2x$	1.14 $10x+z-5y-4x+3z$
1.15 $ab+ab^2+a^2b+3ab$	1.16 $x^2+3y+3x^2-y+x$
1.17 $a^2b+3ab^2+2a^2b-ab^2$	1.18 $5x+y+5xy+6yx+3y$

Tip

Keep things separated and ordered ! You can also try to manipulate terms one after the other. A good rule is to:

- order terms by order, e.g. $x + x^3 + x^2 \rightarrow x + x^2 + x^3$
- order term by alphabetical order, e.g. $a + x + z + x^2 + ab \rightarrow a + x + z + ab + x^2$

It avoids typos and mistakes.

II b) Multiplication

When multiplying algebraic terms, we just multiply everything together, e.g.,

$$2x \times 3y = 6xy$$

we multiply the 2 and the 3 to get the 6.

Note that $6xy$ can be written a number of different ways:

- $6xy$ Usually the best, alphabetical order for x and y
- $x6y$ meh
- $xy6$ this one is really ugly
- $6yx$ Number first is always a good idea
- $y6x$ meh
- $yx6$ this one is really ugly

none of which are wrong, but some are nicer than others!

Note also that xy is $x \times y$, and $x^2y = x \times x \times y$ and $xy^2 = x \times y \times y$ are not the same.

Main Example

The multiplication is illustrated in the following exemples:

1. $3x \times 5y = 15xy$
2. $4x \times y \times 2x = 8xy$ note that $x \times x = x^2$ (x squared)
3. $3x \times 2y \times 4x^2 = 24xy$ note that $3 \times 2 \times 4 = 24$ and $x \times x^2 = x^3$ (x cubed)
4. $x^2 \times 2y^3 \times 5x^3 = 10x^5y^3$ note that $x^2 \times x^3 = x^5$ (x to the power 4)
5. $2x \times (-3y) = -6xy$ note that the negative $3y$ makes the whole product negative
6. $5x \times (-3y) \times (-3z) = 45xyz$ note that the two minuses make a plus

Be careful and multiply things in the correct order ! For instance, $2x - 3y$ and $6x - y$, even if $6 = 2 \times 3$, are completely different things!

Let's check. If $x = 2$ and $y = 1$, then:

$$2x - 3y = 2 \times 2 - 3 \times 1 = 1$$

and

$$6x - y = 6 \times 2 - 1 \times 1 = 11$$

The results are really different !

Exercise 2.

Simplify all these expressions:

2.1 $2x \times 3x^2$	2.2 $2x \times 2y$	2.3 $6x^2 \times 5y$
2.4 $3x \times (-2y)$	2.5 $(-4x) \times (-3y)$	2.6 $12x \times 2y^3 \times 2$
2.7 $a \times 2a \times 4b$	2.8 $12x \times x^3$	2.9 $(-2x) \times (-3x) \times (-4y)$
2.10 $10a \times 2b \times 3c$	2.11 $a \times a \times 2a$	2.12 $5y^2 \times y^2$
2.13 $(-2x^2) \times (-2x)$	2.14 $4x^5 \times 5x^3$	2.15 $a^2 \times 3a^4$
2.16 $-3xy \times 4xy^2$	2.17 $3 \times 2x \times 4xy$	2.18 $(-2x)^2$
2.19 $(-2x)^3$	2.20 $(-2x)^4$	

II c) Division

When dividing algebraic terms the only thing that can be done sometimes is cancelling. This can only be done if there are no addition or subtraction signs, e.g., in $\dfrac{2x^2y}{xy^3}$, some term, but in $\dfrac{2x^2+y}{x+y^3}$, the terms can not be directly cancelled.

It is possible, in some cases, to cancel some terms even if addition or subtraction signs are present, if we can change the expression into a product. It will be shown in an exemple later.

Main Example

Let's simplify $\frac{12x^2y}{6xy^3}$.
There are two x's ($x^2 = x \times x$) in the numerator (top) and one in the denominator (bottom). The one in the denominator is cancelled with one of the two in the numerator:

$$\begin{aligned} \frac{12x^2y}{6xy^3} &= \frac{12x \times xy}{6xy^3} \\ &= \frac{12x \times \cancel{x}y}{6\cancel{x}y^3} \\ &= \frac{12xy}{6y^3} \end{aligned}$$

There are three y's in the denominator ($y^3 = y \times y \times y$) and one in the numerator. It means that one of the three in the denominator can be cancelled with the one in the numerator. It will leave two y in the denominator

$$\begin{aligned} \frac{12xy}{6y^3} &= \frac{12xy}{6y \times y^2} \\ &= \frac{12x \times \cancel{y}}{6\cancel{y}y^2} \\ &= \frac{12x}{6y^2} \end{aligned}$$

The 6 in the denominator goes twice in to the 12 in the numerator.

$$\begin{aligned} \frac{12x}{6y^2} &= \frac{6 \times 2x}{6y^2} \\ &= \frac{\cancel{6} \times 2x}{\cancel{6}y^2} \\ &= \frac{2x}{y^2} \end{aligned}$$

This can be looked at another way by separating numerator and denominator in to single entities, e.g. by writing x^2 as $x \times x$ and so on:

$$\begin{aligned} \frac{12x^2y}{6xy^3} &= \frac{6 \times 2 \times x \times x \times y}{6 \times x \times y \times y \times y} \\ &= \frac{\cancel{6} \times 2 \times \cancel{x} \times x \times \cancel{y}}{\cancel{6} \times \cancel{x} \times y \times y \times \cancel{y}} \\ &= \frac{2x}{y^2} \end{aligned}$$

A few other illustrations:

$$\begin{aligned} \frac{7x^3yz^4}{14xyz^3} &= \frac{7 \times x \times x \times x \times y \times z \times z \times z \times z}{14 \times x \times y \times z \times z \times z} \\ &= \frac{x^2z}{2} \end{aligned}$$

We usually can not simplify if there is additions or subtraction.
For instance, you can NOT do the following:

$$\frac{2x+1}{2} = \frac{\cancel{2}x+1}{\cancel{2}}, \text{THIS IS WRONG}$$

To verify it, if $x = 1$, we have $\frac{2x+1}{2} = \frac{2 \times 1 + 1}{2} = \frac{3}{2}$, and after the wrong simplification, we would have

$\frac{\cancel{2}\times 1+1}{\cancel{2}} = \frac{1}{2}$, and the two results are different.

Nevertheless, if we can rewrite the expression as a product:

$$\frac{2x^2-3x}{x+4x^2} = \frac{x(2x-3)}{x(1+4x)} = \frac{(2x-3)}{(1+4x)}$$

We can simplify. Here, the key point is to make sure that the main form of the expression is a product in the numerator $x(2x-3)$ and denominator $x(1+4x)$. The x is there present in top and bottom, we can simplify.

Exercise 3.

Please simplify:

3.1	$\frac{3xyz}{xz}$	3.2	$\frac{16xy^2z^3}{4xyz}$	3.3	$\frac{4x^3yz}{12xy^2}$
3.4	$\frac{14xyz^3}{2yz}$	3.5	$\frac{6ab^2}{2b}$	3.6	$\frac{12x^2yz}{6xy}$
3.7	$(16xy^2)\div(4y)$	3.8	$(20abc)\div(5b^2)$	3.9	$(14x^2y^3z)\div(7xy^2)$
3.10	$(12xy)\div(2xy)$	3.11	$\frac{52ab^4cd^4}{13bd^2}$	3.12	$\frac{12a+b}{6a}$

III Removing brackets

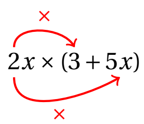
III a) Single brackets

III a) i The rule

To remove brackets , follow this rule:

⇝ multiply everything inside the brackets by the term outside the brackets

For instance, with the expression $2x(3+5x)$ we must multiply both the 3 by the $2x$ and the $5x$ by the $2x$:



It results in:

$$2x(3+5x) = 2x\times 3 + 2x\times 5x = 6x + 10x^2$$

Here we have multiplied the $2x$ by 3 to get $6x$ and we have multiplied the $2x$ by $5x$ to get $10x^2$.

$$3x(4x+9) = 3x\times 4x + 3x\times 9 = 12x^2 + 27x$$

$$5x(3x-6) = 5x\times 3x + 5x\times(-6) = 15x^2 - 30x$$

Note that $5x$ multiplied by -6 gives us a negative quantity i.e. $-30x$

$$-x^2(3x-5x^3) = -x^2\times 3x + (-x^2)\times(-5x)$$

Note that the first term is a negative times a positive which is a negative and the second term is a negative times a negative which gives a positive.

Tip

$-x^2(3x-5x^3) = -3x^3+5x^5$ is correct.
But it is usually written as $5x^5-3x^3$. Putting the positive term first makes things easier to read, and you are less prone to forget a "-" sign.

Main Example

Let's expand $5x(x+4)-6(2x-3)$.
First, let's look at the first $5x(x+4)$.

$$\begin{aligned} 5x(x+4) &= 5x\times x+5x\times 4 \\ &= 5x^2+20x \end{aligned}$$

then the second term $-6(2x-3)$.

$$\begin{aligned} -6(2x-3) &= -6\times 2x+(-6)\times(-3) \\ &= -12x+18 \\ &= 18-12x \end{aligned}$$

We can now add them:

$$\begin{aligned} 5x(x+4)-6(2x-3) &= (5x^2+20x)+(18-12x) \\ &= 5x^2+8x+18 \end{aligned}$$

Let's have a few other exemples:

Exercise 4.
Expand, remove the brackets and simplify the following expressions:

4.1 $2(x+3)$

4.2 $3(x+4)$

4.3 $4(x-5)$

4.4 $2(3-x)$

4.5 $2x(x+3)$

4.6 $x(x-3)$

4.7 $-3x(x+5)$

4.8 $-5x(x-5)$

4.9 $-5x(-x^2-3x)$

4.10 $4x(x^3+3x+4)$

4.11 $3(x+1)+2(3x-4)$

4.12 $5(2a+4)-3(4a+2)$

4.13 $4(1-2x)-3(3x-4)$

4.14 $2x(x-5)-x(x-2)-3x(x-5)$

4.15 $3(a-b)-2(2a-3b)+4(a-3b)$

4.16 $3x(x^2+4)-2x(2x^2-3)-3(x^2+5)$

III b) Double Brackets

III b) i The rule

In expressions with double brackets, we multiply every term in the first bracket with every term in the second bracket. For instance, in $(x+2)\times(y+6)$ we multiply the x with the y and the x with the 6. Them We multiply the 2 with the y and the 2 with the 6:

$$(x+2)\times(y+6) = \underbrace{x\times y+x\times 6}_{x(y+6)\text{: first term of first bracket}} + \underbrace{2\times y+2\times 6}_{2(y+6)\text{: second term of first bracket}}$$

Main Example

Let's simplify

$$(x+4)(x-3)$$

A first option is to split the first brackets:

$$(x+4)(x-3) = \underbrace{x}_{\text{from } x+4}(x-3) + \underbrace{4}_{\text{from } x+4}(x-3)$$

and then to apply the rules from Sec. III a):

- ➔ $x(x-3) = x^2 - 3x$
- ➔ $4(x-3) = 4x - 12$

and, consequently:

$$(x+4)(x-3) = x^2 - 3x + 4x - 12 = x^2 + x - 12$$

A second option is to take one element of the first bracket after the other, and to multiply it by each element of the second bracket:

- ➔ the x from $(x+4)$ is multiplied by:
  - the x from $(x-3)$: x^2
  - the -3 from $(x-3)$: $-3x$
- ➔ the 4 from $(x+4)$ is multiplied by:
  - the x from $(x-3)$: $4x$
  - the -3 from $(x-3)$: -12

and, consequently:

$$(x+4)(x-3) = x^2 - 3x + 4x - 12 = x^2 + x - 12$$

Let's have a few other exemples:

$$\begin{aligned}(x-4)(x-7) &= x \times x + x \times (-7) - 4 \times x - 4 \times (-7)\\ &= x^2 - 7x - 4x + 28 \text{ note that -4} \times \text{(-7) is positive}\\ &= x^2 - 11x + 28\end{aligned}$$

$$\begin{aligned}(2x+3)(x-5) &= 2x \times x + 2x \times (-5) + 3 \times x + 3 \times (-5)\\ &= 2x^2 - 10x + 3x - 15\\ &= x^2 - 7x - 15\end{aligned}$$

$$\begin{aligned}(2x-3)(5x-7) &= 2x \times 5x + 2x \times (-7) - 3 \times 5x - 3 \times (-7)\\ &= 10x^2 - 14x - 15x + 21\\ &= x^2 - 29x + 21\end{aligned}$$

Exercise 5.

Expand, remove the brackets and simplify the following expressions:

5.1	$(x+1)(x+2)$	5.2	$(x+3)(x+1)$	5.3	$(2x+1)(x-1)$
5.4	$(x-1)(x-4)$	5.5	$(x+3)(x-3)$	5.6	$(x+1)^2$
5.7	$(x-2)^2$	5.8	$(2x+3)^2$	5.9	$(2u+v)(u-v)$

5.10 $(x-y)^2$ 5.11 $(2x+1)(2x-1)$ 5.12 $(x+y)^2$

5.13 $(x+y)(x-y)$ 5.14 $(x+y)^3$

IV Notes on parenthesis

Parenthesis can be tricky.

You already know that $5(a+b) = 5a+5b$.
And also that: $5(ab) = 5ab$
You do know about functions ? For instance, $\sin(a+b)$?
Well, be carefull. It is not the same kind of parenthesis!

$$(a+b) = a+b$$

but

$$\sin(a+b) \neq sina + sinb$$

In $(a+b) = a+b$, we use the parenthesis for grouping terms. In $\sin(a+b)$, we use the parenthesis for showing what we give to the function sin.

But it is not that simple !
Some functions are *linear*. It means, that for these kind of very specific functions,

$$f(a+b) = f(a) + f(b)$$

Quite confusing...

But it is generally not the case, so please pay attention to the kind of parenthesis you are encountering !

V Solutions to exercises

Solution 1.

1.1 $7x+4x-x=10x$

1.2 $3x-2+x=4x-2$

1.3 $5x+3x+6=8x+6$

1.4 $x-3x+5x+12x=15x$

1.5 $2y+5y+3y-y=9y$

1.6 $y-3y+6+4y=2y+6$

1.7 $2a+3b+4a-3b=6a$

1.8 $3x-(-4x)+5x+2=12x+2$

1.9 $5x-2x-(-4x)-7x=0$

1.10 $8x-3x+2y-(-x)+(-5y)=6x-3y$

1.11 $7x-9-3x+x-10x=-5x-9$

1.12 $x^2+3x^2-5y+y^2=x^2+y^2-5y$

1.13 $3y-3x+4-y-2x=-5x-2y+4$

1.14 $10x+z-5y-4x+3z=6x-5y+z$

1.15 $ab+ab^2+a^2b+3ab=a^2b+ab^2+4ab$

1.16 $x^2+3y+3x^2-y+x=4x^2+x+yy^2+2y$

1.17 $a^2b+3ab^2+2a^2b-ab^2=3a^2b+2ab^2$

1.18 $5x+y+5xy+6yx+3y=11xy+5x+4y$

Solution 2.

2.1 $2x \times 3x^2 = 6x^3$

2.2 $2x \times 2y = 4xy$

2.3 $6x^2 \times 5y = 30x^2y$

2.4 $3x \times (-2y) = -6xy$

2.5 $(-4x) \times (-3y) = 12xy$

2.6 $12x \times 2y^3 \times 2 = 48xy^3$

2.7 $a \times 2a \times 4b = 8a^2b$

2.8 $12x \times x^3 = 12x^4$

2.9 $(-2x) \times (-3x) \times (-4y) = -24x^2y$

2.10 $10a \times 2b \times 3c = 60abc$

2.11 $a \times a \times 2a = 2a^3$

2.12 $5y^2 \times y^2 = 5y^4$

2.13 $(-2x^2) \times (-2x) = 4x^3$

2.14 $4x^5 \times 5x^3 20x^8$

2.15 $a^2 \times 3a^4 = 3a^6$

2.16 $-3xy \times 4xy^2 = -12x^2y^3$

2.17 $3 \times 2x \times 4xy = 24x^2y$

2.18 $(-2x)^2 = 4x^2$

2.19 $(-2x)^3 = -8x^3$

2.20 $(-2x)^4 = 16x^4$

Solution 3.

3.1 $\dfrac{3xyz}{xz} = 3y$

3.2 $\dfrac{16xy^2z^3}{4xyz} = 4yz^2$

3.3 $\dfrac{4x^3yz}{12xy^2} = \dfrac{x^2z}{3y}$

3.4 $\dfrac{14xyz^3}{2yz} = 7xz^2$

3.5 $\dfrac{6ab^2}{2b} = 3ab$

3.6 $\dfrac{12x^2yz}{6xy} = 2xz$

3.7 $(16xy^2) \div (4y) = 4xy$

3.8 $(20abc) \div (5b^2) = \dfrac{4ac}{b}$

3.9 $(14x^2y^3z) \div (7xy^2) = 2xyz$

3.10 $(12xy) \div (2xy) = 6$

3.11 $\dfrac{52ab^4cd^4}{13bd^2} = 4ab^3cd^2$

3.12 $\dfrac{12a+b}{6a} = \dfrac{12a+b}{6a}$

Solution 4.

4.1 $2(x+3) = 2x+6$

4.2 $3(x+4) = 3x+12$

4.3 $4(x-5) = 4x-20$

4.4 $2(3-x) = 6-2x$

4.5 $2x(x+3) = 2x^2+6x$

4.6 $x(x-3) = x^2-3x$

4.7 $-3x(x+5) = -3x^2-15x$

4.8 $-5x(x-5) = -5x^2+25x$

4.9 $-5x(-x^2-3x) = 5x^3+15x^2$

4.10 $4x(x^3+3x+4) = 4x^4+12x^2+16x$

4.11 $3(x+1)+2(3x-4) = 9x-5$

4.12 $5(2a+4)-3(4a+2) = 14-2a$

4.13 $4(1-2x)-3(3x-4) = -17x-8$

4.14 $2x(x-5)-x(x-2)-3x(x-5) = -2x^2-7x$

4.15 $3(a-b)-2(2a-3b)+4(a-3b) = -a-9b$

4.16 $3x(x^2+4)-2x(2x^2-3)-3(x^2+5) = -x^3-3x^2+6x-15$

Solution 5.

5.1 $(x+1)(x+2) = x^2+3x+2$

5.2 $(x+3)(x+1) = x^2+4x+3$

5.3 $(2x+1)(x-1) = 2x^2-x-1$

5.4 $(x-1)(x-4) = x^2-5x+4$

5.5 $(x+3)(x-3) = x^2-9$

5.6 $(x+1)^2 = x^2+2x+1$

5.7 $(x-2)^2 = x^2-4x+4$

5.8 $(2x+3)^2 = 4x^2+12x+9$

5.9 $(2u+v)(u-v) = 2u^2-uv-v^2$

5.10 $(x-y)^2 = x^2-2xy+y^2$

5.11 $(2x+1)(2x-1) = 4x-1$

5.12 $(x+y)^2 = x^2+2xy+y^2$

5.13 $(x+y)(x-y) = x^2-y^2$

5.14 $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

Bibliography

[Croft and Davidson, 2016] Croft, A. and Davidson, R. (2016). *Foundation Maths.* Pearson.