Mathematics for Engineers–ENG 3009, 2018-2019

Introduction to formulas

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As always, please free to refer to the book [Croft and Davidson, 2016] for details.

I Introduction

This unit covers evaluation of formulae (or substituting into formulae) and transposition (or rearrangement) of formulae.

II Evaluating Formulae

Def.

SUBSTITUTION: For substituting a value in an equation, all the occurances of the concerned variable in the equation has to be substituted with the value.

Main Example

For example if we are given that $y = mx + c$ and we know that $m = 2$, $x = 5$ and $c = 4$ then we can substitute these in to the formula:

$x = 5$

$$y = mx + c \rightsquigarrow y = 2 \times 5 + 4$$

$m = 2$ $c = 4$

Now the rules of order of operations tell us that we do multiplication before addition:

$$\begin{aligned} y &= 2 \times 5 + 4 \\ &= 10 + 4 \\ &= 14 \end{aligned}$$

It is wrong to add the 5 and the 4 first: You would get $5 + 4 = 9$ and then $2 \times 9 = 18 \neq 14$. This would be correct if the 5 and the 4 are in brackets : $y = m \times (x + c)$. In this case, they are not.

III Order of operations

The rules of order of operations are important. You can not do proper computations without them.

A mnemonic way is to remember BODMAS.

Def.

BODMAS: Bodmas is a way of remembering that you do brackets first, then any multiplication or division and then any addition or subtraction last.

- ➔ Brackets
- ➔ Of
- ➔ Division
- ➔ Multiplication
- ➔ Addition
- ➔ Subtraction

Also, remember it is left to right !

Main Example

Why is it so important ? Think of

$$6 \div 2(1+2)$$

Is it equal to 1, or to 9 ?
So

$$\begin{aligned} 6 \div 2(1+2) &= 6 \div 2 \times 3 && \text{first deal with the brackets, } (1+2)=3 \\ &= 3 \times 3 && \text{then division, } 6 \div 2 = 3\text{, from the left !} \\ &= 9 \end{aligned}$$

without following the rules, it is easy to do first $2(1+2) = 6$, and hence think that $6 \div 2(1+2)$ is equal to $6 \div 6$, or to have $6 \div 2 \times 3$, and multiply from the right first, and have once again $6 \div 6$. Remember BODMAS !

The following examples demonstrate this and show some of the pitfalls that students fall in to when evaluating formulae.

If $y = 3x$, find y when:

- ➔ $x = 1$:
  $x = 1$ means that $y = 3 \times 1 = 3$
- ➔ $x = 0$:
  $x = 0$ means that $y = 3 \times 0 = 0$
- ➔ $x = -4$:
  $x = 4$ means that $y = 3 \times (-4) = -12$

If $y = 3x + 4$, find y when:

- ➔ $x = 4$:
  $x = 4$ means that $y = 3 \times 4 + 4 = 12 + 4 = 16$
- ➔ $x = 0$:
  $x = 0$ means that $y = 3 \times 0 + 4 = 4$
- ➔ $x = -2$:
  $x = -2$ means that $y = 3 \times (-2) + 4 = -6 + 4 = 2$

If $y = 3x^2$, find y when:

- ➔ $x = 1$:
  $x = 1$ means that $y = 3 \times 1 \times 1 = 1$
- ➔ $x = 0$:
  $x = 0$ means that $y = 3 \times 0 \times 0 = 0$
- ➔ $x = -2$:
  $x = -2$ means that $y = 3 \times (-2) \times (-2) = 3 \times 4 = 12$

Note that $3x^2 = 3 \times x \times x$, i.e. the x is squared but not the 3. It is a common error to multiply the 3 by the x and then square the result, and this is wrong. You should square the x first and then multiply by the 3.

If $y = x^3 - x$, find y when:

- ➔ $x = 1$:
  $x = 1$ means that $y = 1^3 - 1 = 0$
- ➔ $x = 3$:
  $x = 3$ means that $y = 3^3 - 3 = 27 - 3 = 24$
- ➔ $x = -2$:
  $x = -2$ means that $y = (-2)^3 - (-2) = -8 + 2 = -6$
  Note that $(-2) \times (-2) \times (-2) = -8$ as there is 3 minus signs

➔ $x = -4$:
$x = -4$ means that $y = (-4)^3 - (-4) = -64 + 4 = -60$

If $y = \frac{1}{x^2} - x^3$, find y when:

➔ $x = 1$:
$x = 1$ means that $y = \frac{1}{1^2} - 1^3 = 1 - 1 = 0$

➔ $x = 2$:
$x = 2$ means that $y = \frac{1}{2^2} - 2^3 = \frac{1}{4} - 8 = -\frac{31}{4}$

➔ $x = 0$:
$x = 2$ means th at $y = \frac{1}{0^2} - 0^3$. This is not defined, because $\frac{1}{0}$ is not defined.

➔ $x = -2$:
$x = -2$ means that $y = \frac{1}{(-2)^2} - (-2)^3 = \frac{1}{4} - (-8) = \frac{33}{4}$

Exercise 1.

1.1 If $y = 2x - 2$, find y when

a $x = 1$

b $x = 0$

c $x = -3$

1.2 If $y = 3x + 5$, find y when

a $x = 1$

b $x = 0$

c $x = -3$

1.3 If $y = 2x^2 + 2$, find y when

a $x = 1$

b $x = -1$

c $x = 2$

1.4 If $y = 3x^2 - x$, find y when

a $x = 1$

b $x = 2$

c $x = \frac{1}{3}$

d $x = -2$

1.5 If $y = x^3 - 2x^2$, find y when

a $x = 3$

b $x = 0$

c $x = -2$

1.6 If $V = \frac{4}{3}\pi r^3$, find V when
$r = 2.5$cm (to 3 decimal places)

1.7 If $V = \pi r^2 h$, find V when
$r = 3.6$cm and $h = 8$cm (to 3 significant figures)

1.8 If $y = x^2 - 4x + 2$, find y when

a $x = 0$

b $x = 1$

c $x = 2$

d $x = -2$

1.9 If $A = 4\pi r^2$, find A (to 3 significant figures) when

a $r = 0.1$

b $r = 1$

c $r = 10$

1.10 If $V = \frac{\pi h}{3}(R^2 - r^2)$, find V (to 3 significant figures), given $h = 14$, when

a $R = 4$ and $r = 2$

b $R = 8$ and $r = 4$

1.11 If $Q = mc(t_2 - t_1)$, find Q when $m = 10, t_1 = 50$, $t_2 = 150$ and $c = 500$.

1.12 If $P = \frac{V^2}{R}$, find P when $V = 12$ and $R = 10$.

1.13 If $R = \frac{R_1 R_2}{R_1 + R_2}$, find R when

a $R_1 = 5$ and $R_2 = 20$

b $R_1 = 5$ and $R_2 = -1$

1.14 If $A = \pi r^2$, find A when $r = 3.6$ (up to 2 decimal places).

1.15 If $V = \frac{1}{3}\pi r^2 h$, find V when $r = 4$ and $h = 1.2$ (up to 4 significant figures).

1.16 If $L = \frac{412}{W^2}$, find L when $W = 14$, (up to 2 significant figures).

1.17 If $z = \sqrt{2 + ac}$, find z (up to 3 decimal places) when

a $a = 4$ and $c = 8.5$

b $a = 4$ and $c = 0.5$

c $a = 4$ and $c = -0.5$

d $a = 4$ and $c = -1$

1.18 If $P = \rho g h$, find P when $\rho = 1000$, $g = 9.8$ and $h = 4$.

1.19 If $z = \sqrt{\frac{(xy)^2}{u + v}}$, find z when $x = 2$, $y = 4$, $u = 6$ and $v = 2$ (up to 3 decimal places).

1.20 If $k = ac + bc - c^2$, find k when $a = 4$, $b = 6.3$ and $c = -2$.

IV Transposition of Formulas

IV a) Equations of the form $y = mx$

In mathematics and science it is often necessary to manipulate formulas. For example the equation of a straight line passing through the origin is

$$y = mx$$

and y can be found as long as m and x are known. Conversely, if y and m are known, it is possible to find x.

Def.

SOLUTION OF A LINEAR EQUATION:
It is achieved by

- ➔ rearranging the formula to make x the subject
- ➔ substituting the known values
- ➔ compute x

1. and 2. can be swapped.

If $y = 10$ and $m = 5$, then we have $10 = 5x$. Divinding both sides with 5 gives:

$$\frac{10}{5} = \frac{5x}{5} \Rightarrow x = 2$$

Remember you can do anything to one side of an equation provided you do the same to the other side, in this example we divided both sides by 5.

Going further

You can not divide both sides by 0 ! If you do, you can "show" weird things. For instance, think of two numbers x and y, that are equal and not equal to zeros:

$$y = x, x \neq 0, y \neq 0$$

then, multiplying both sides with x:

$$x^2 = xy$$

substract x^2 from both sides:

$$x^2 - y^2 = xy - y^2$$

Or, note that $x^2 - y^2 = (x + y) \times (x - y)$ and $xy - y^2 = y \times (x - y)$. Dividing both sides by $x - y$ (because it is 0, it should not be done !!):

$$\begin{aligned} \frac{x^2 - y^2}{x - y} &= x + y && \text{left side} \\ &= \frac{xy - y^2}{x - y} && \text{right side} \\ &= y \end{aligned}$$

so we have $x + y = y$. Because $x = y$, it means that $2y = y$. Because $y \neq 0$, it means, by dividing by y, that $2 = 1$.

Pretty wrong.

And it is just because we divided with 0.

Given that

$$y = mx$$

we rearrange this to make x the subject:

$$\frac{y}{m} = \frac{mx}{m} \Rightarrow x = \frac{y}{m}$$

For instance, we have in electronics the Ohms law:

$$\mathrm{V} = \mathrm{IR}$$

If we want to have R as the subject:

$$\mathrm{R} = \frac{\mathrm{V}}{\mathrm{I}}$$

Main Example

Another interesting law is the law of perfect gas. The pressure varies with the inverse of the volume:

$$\mathrm{P} = \frac{\mathrm{C}}{\mathrm{V}}$$

To get the constant C, we multiply both sides by V. It will cancel it out on the right hand side:

$$\mathrm{P} = \frac{\mathrm{C}}{\mathrm{V}} \Leftrightarrow \mathrm{PV} = \frac{\mathrm{CV}}{\mathrm{V}},$$

and, cancelling the V's

$$\mathrm{PV} = \mathrm{C}$$

If we want to have get the volume, it might be a bit more tricky, as V is in the denominator. In $\mathrm{P} = \frac{\mathrm{C}}{\mathrm{V}}$, we might want want to divide first by C, which leads to

$$\frac{\mathrm{P}}{\mathrm{C}} = \frac{\mathrm{C}}{\mathrm{CV}} = \frac{1}{\mathrm{V}}$$

When you simplify, V remains in the denominator, you can NOT write:

$$\frac{\mathrm{P}}{\mathrm{C}} = \frac{\mathrm{C}}{\mathrm{CV}} = \mathrm{V}, \text{Wrong !!}$$

Note that you can flip upside down the equations, as there is fraction on both sides:

$$\frac{a}{c} = \frac{b}{d} \Leftrightarrow \frac{c}{a} = \frac{d}{b}$$

But

$$\frac{a}{c} + 1 = \frac{b}{d} \Leftrightarrow \frac{c}{a} + 1 = \frac{d}{b}, \text{Wrong !!}$$

To check it, simply multiply both side with cd and divide both side with ab.

Exercise 2.
Transpose the following

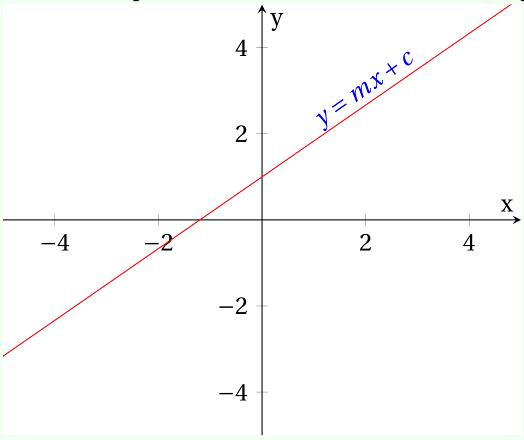
2.1 $\mathrm{F} = ma$ for a	2.2 $v = rh$ for r	2.3 $x = \frac{y}{b}$ for y
2.4 $c = \pi d$ for d	2.5 $s = \pi dn$ for n	2.6 $pv = c$ for v
2.7 $x = \frac{a}{y}$ for a	2.8 $x = \frac{a}{y}$ for y	2.9 $\mathrm{I} = \frac{\mathrm{E}}{\mathrm{R}}$ for R
2.10 $x = \frac{u}{a}$ for u	2.11 $\mathrm{P} = \frac{\mathrm{RT}}{\mathrm{V}}$ for T	2.12 $\mathrm{S} = \frac{tv}{k}$ for k

IV b) Equation of the form $y = mx + c$

The equation of a straight line is $y = mx + c$.

Def.

STRAIGHT LINE: It corresponds to a linear function whose graph is a line in the plane.



It can be described by the following equation:

$$y = mx + b$$

Let's assume that m and c are known.

It means that for any x, we can report, on the axis, one y and, consequently, draw a line. By nature, the line is straight and the name follows. More detailed will be provided in the chapter dedicated to straight lines.

y can be found if m, x and c are known. The values just have to be substituted.

Conversely, if y, m and c are known, it is possible to find x. For that, one can rearrange the formula to make x the subject, and then substitute the known values to figure x out. You can also first substitute the values, and then rearrange the terms.

Main Example

Let's consider

$$y = mx + c$$

If $y = 10$, $m = 2$ and $c = 4$, we have:

$$10 = 2x + 4$$

Subtracting 4 from both sides leads to:

$$10 - 4 = 2x + 4 - 4 \Leftrightarrow 6 = 2x$$

Now dividing both sides by 2 gives:

$$\frac{6}{2}\frac{\cancel{2}x}{\cancel{2}} \Leftrightarrow x = 3$$

It is also possible to rearrange the terms first !

$$\begin{aligned} y = mx + c \quad &\Leftrightarrow \quad y - c = mx + \cancel{c} - \cancel{c} && \text{subtracting c} \\ &\Leftrightarrow \quad \frac{y-c}{m} = \frac{\cancel{m}x}{\cancel{m}} && \text{dividing by m} \\ &\Leftrightarrow \quad x = \frac{y-c}{m} \end{aligned}$$

We can now susbtitute the known values:

$$x = \frac{10-4}{2} = 3$$

But we can also rearrange the term in a different way !

$$\begin{aligned} y = mx + c \quad &\Leftrightarrow \quad \frac{y}{m} = \frac{\cancel{m}x}{\cancel{m}} + \frac{c}{m} && \text{dividing by m} \\ &\Leftrightarrow \quad \frac{y}{m} - \frac{c}{m} = x + \cancel{\frac{c}{m}} - \cancel{\frac{c}{m}} && \text{subtracting } c/m \\ &\Leftrightarrow \quad x = \frac{y}{m} - \frac{c}{m} \end{aligned}$$

We can now susbtitute the known values:

$$x = \frac{10}{2} - \frac{4}{2} = 3$$

As expected, the two methods have the same results. Indeed, we have $\frac{y}{m} - \frac{c}{m} = \frac{y-c}{m}$.

Tip

It is often thought that, when rearranging formulas, there is only way to go about it, and if you don't choose this particular route then you will get it wrong.

This is not true!

You can use any method you want providing that you do not break any of the rule. Basically, you must remember to always do the same to both sides of the equation.

Let's have a few other examples. Given $v = u + at$.

➔ if we want to make u the subject:

$$\begin{aligned} v = u + at \quad &\Leftrightarrow \quad v - at = u + \cancel{at} - \cancel{at} && \text{subtracting } at \\ &\Leftrightarrow \quad u = v - at \end{aligned}$$

➔ if we want to make t the subject:

$$\begin{aligned} v = u + at \quad &\Leftrightarrow \quad v - u = \cancel{u} - \cancel{u} + at && \text{subtracting } u \\ &\Leftrightarrow \quad \frac{v-u}{a} = \frac{\cancel{a}t}{\cancel{a}} && \text{dividing by } a \\ &\Leftrightarrow \quad t = \frac{v-u}{a} \end{aligned}$$

Note that the second step can NOT be:

$$v - u = at \Leftrightarrow \frac{v}{a} - u = \frac{at}{a} \text{ not} \quad \text{dividing everything by } a$$

Here we are not dividing everything by a (the u), consequently the equivalence is not true.

Exercise 3.

Transpose the following

3.1 $n = \mathrm{P} - 14.7$, for P

3.2 $n = p + cr$, for p

3.3 $n = p + cr$, for c

3.4 $v = u + at$, for a

3.5 $y = ax + b$, for x

3.6 $y = \frac{x}{5} + 17$, for x

3.7 $a = b - cx$, for x

3.8 $\mathrm{D} = \mathrm{B} - 1.2d$, for d

3.9 $\mathrm{V} = \frac{2\mathrm{R}}{\mathrm{R} - r}$, for r

3.10 $\mathrm{C} = \frac{\mathrm{E}}{\mathrm{R} + r}$, for E

3.11 $\mathrm{S} = \pi r(r + h)$, for h

3.12 $\mathrm{H} = ws(\mathrm{T} - t)$, for T

3.13 $\mathrm{C} = \frac{\mathrm{N} - n}{2\mathrm{P}}$, for N

3.14 $\mathrm{T} = \frac{12(\mathrm{D} - d)}{\mathrm{L}}$, for d

3.15 $\mathrm{S} = \pi r^2 h + 2\pi r$, for h

IV c) Equations with squares and square root

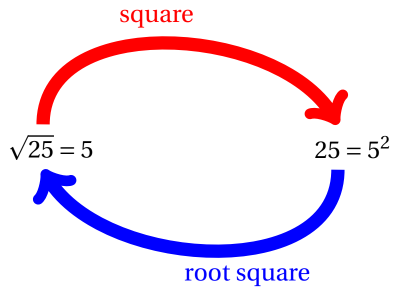
Equations with square roots or squares in are quite common in engineering. The same approach is used as with previous equations.

Tip

Remember you can do anything to one side of an equation provided you do the same to the other side.

To remove a square root you must square both sides and the square root will cancel out (square and square root are the inverse of each other, see the chapter *Introduction to indices and logs*).

Similarly if you wish to remove a square you square root both sides.



Tip

Be careful, you have to apply actions on the whole sides ! If not, the results will mostly be incorrect. For instance, $u + v = \sqrt{w}$ It is tempting to think that $w = \sqrt{w}^2 = u^2 + v^2$ but this is wrong. Indeed, if $u = 1, v = 2$, then $u + v = 3$ and then $\sqrt{w} = 3$, meaning $w = 9$. But

$$u^2 + v^2 = 1^2 + 2^2 = 5 \neq 9$$

It is annoying, but apply the power to the *whole* side.

Main Example

➔ Given that $v = \sqrt{2gh}$, let's make h the subject:

$$\begin{aligned} v = \sqrt{2gh} \quad &\Leftrightarrow \quad v^2 = (\sqrt{2gh})^2 && \text{square both sides} \\ &\Leftrightarrow \quad v^2 = 2gh && \text{no more } \sqrt{} \\ &\Leftrightarrow \quad h = \frac{v^2}{2g} && \text{divide both sides with } 2g \end{aligned}$$

➔ Given that $v^2 = u^2 + 2as$, let's make u the subject:

$$\begin{aligned} v^2 = u^2 + 2as \quad &\Leftrightarrow \quad v^2 - 2as = u^2 + \cancel{2as} - \cancel{2as} && \text{subtract } 2as \text{ on both sides} \\ &\Leftrightarrow \quad v^2 - 2as = u^2 \\ &\Leftrightarrow \quad u = \sqrt{v^2 - 2as} && \text{square root both sides} \end{aligned}$$

Note that if you square root first like $\sqrt{v^2} = \sqrt{u^2 - 2as}$, it only make things worse, as you cannot square root (or square) elements individually (remember that $\sqrt{v^2} \neq \sqrt{u^2} + \sqrt{2as}$).

➔ Given that $s = 4\pi r^2$, let's make r the subject:

$$\begin{aligned} s = 4\pi r^2 \quad &\Leftrightarrow \quad \sqrt{s} = \sqrt{4\pi r^2} && \text{square root both sides} \\ &\Leftrightarrow \quad \sqrt{s} = 2\sqrt{\pi} r \\ &\Leftrightarrow \quad r = \frac{\sqrt{s}}{2\sqrt{\pi}} && \text{divide both sides with } 2\sqrt{\pi} \end{aligned}$$

Note that $\frac{\sqrt{s}}{2\sqrt{\pi}}$ can be rewritten as $\frac{1}{2}\sqrt{\frac{s}{\pi}}$ or as $\sqrt{\frac{s}{4\pi}}$ (do not forget to square an element - here the 2 - before putting it under the square root sign).

We can also do by rearranging terms first:

$$\begin{aligned} s = 4\pi r^2 \quad &\Leftrightarrow \quad \frac{s}{4\pi} = \frac{\cancel{4\pi} r^2}{\cancel{4\pi}} && \text{divide both sides with } 4\pi \\ &\Leftrightarrow \quad \sqrt{\frac{s}{4\pi}} = r && \text{square root both sides} \end{aligned}$$

to get the same result.

Exercise 4.

Transpose the following

4.1 $v = \sqrt{2gh}$, for g

4.2 $z = \sqrt{\frac{x}{x+y}}$, for x

4.3 $w = k\sqrt{d}$, for d

4.4 $P = \frac{mv^2}{r}$, for v

4.5 $t = 2\pi\sqrt{\frac{L}{g}}$, for L

4.6 $v^2 = u^2 - 2as$, for a

4.7 $y = ax^2$, for x

4.8 $r^2 = x^2 + y^2$, for y

4.9 $t = 2\pi\sqrt{\frac{w}{gf}}$, for f

4.10 $c = 2\sqrt{2hr - h^2}$, for r

V Solutions to exercises

Solution 1.

1.1

a $y = 0$
b $y = -2$
c $y = -8$

1.2

a $y = 8$
b $y = 5$
c $y = -4$

1.3

a $y = 4$
b $y = 4$
c $y = 10$

1.4

a $y = 2$
b $y = 10$
c $y = 0$
d $y = 14$

1.5

a $y = 9$
b $y = 0$
c $y = -16$

1.6 V = 65.450

1.7 V = 326

1.8

a $y = 2$
b $y = -1$
c $y = -2$
d $y = 14$

1.9

a A = 0.126
b A = 12.6
c A = 1260

1.10

a V = 176
b V = 704

1.11 Q = 500000

1.12 P = 14.4

1.13

a R = 4
b R = −1.25

1.14 A = 40.72

1.15 V = 20.11

1.16 $L = 2.1$

1.17

a $z = 6$

b $z = 2$

c $z = 0$

d undefined

1.18 $P = 39200$

1.19 $z = 2.828$

1.20 $P = -24.6$

Solution 2.

2.1 $\frac{F}{m} = a$

2.2 $\frac{v}{h} = r$

2.3 $xb = y$

2.4 $d = \frac{c}{\pi}$

2.5 $n = \frac{s}{\pi d}$

2.6 $v = \frac{c}{p}$

2.7 $a = xy$

2.8 $y = \frac{a}{x}$

2.9 $R = \frac{E}{I}$

2.10 $u = xa$

2.11 $T = \frac{PV}{R}$

2.12 $k = \frac{tv}{S}$

Solution 3.

3.1 $P = n + 14.7$

3.2 $p = n - cr$

3.3 $c = \frac{n-p}{r}$

3.4 $a = \frac{v-u}{t}$

3.5 $x = \frac{y-b}{a}$

3.6 $x = 5y - 85$

3.7 $x = \frac{b-a}{c}$

3.8 $d = \frac{B-D}{1.2}$

3.9 $r = R - \frac{2R}{V}$

3.10 $E = C \times (R + r)$

3.11 $h = \frac{S}{\pi r} - r$

3.12 $T = \frac{H}{ws} + t$

3.13 $N = 2PC + n$

3.14 $d = D - \frac{TL}{12}$

3.15 $h = \frac{S - 2\pi}{\pi r^2}$

Solution 4.

4.1 $g = \frac{v^2}{2h}$

4.2 $x = \frac{yz}{1-z}$

4.3 $d = \left(\frac{w}{k}\right)^2$

4.4 $v = \sqrt{\frac{r}{m}(P - mg)}$

4.5 $L = g\left(\frac{t}{2\pi}\right)^2$

4.6 $a = \frac{u^2 - v^2}{2s}$

4.7 $x = \sqrt{\frac{y}{a}}$

4.8 $y = \sqrt{r^2 - x^2}$

4.9 $f = \frac{w}{f}\sqrt{\frac{2\pi}{t}}$

4.10 $r = \frac{1}{2h}\left(\frac{c^2}{4} + h^2\right)$

Bibliography

[Croft and Davidson, 2016] Croft, A. and Davidson, R. (2016). *Foundation Maths.* Pearson.