Mathematics for Engineers–ENG 3009, 2018-2019

Introduction to indices and logs

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As always, please free to refer to the book [Croft and Davidson, 2016] for details.

I Introduction

This unit covers indices and their laws, and logarithms and their uses in solving some types of equations.

II Indices, powers and roots

Def.

INDICES: Indices are powers or exponents. Note that the singular of indices is index. A power, or an index, is used to write a product of a number by itself a certain number of time in a very compact way.

For example 3^4 is read 3 raised to the power of 4. It correponds to 3 multiplied by itself 4 times:

$$3^4 = \underbrace{3 \times 3 \times 3 \times 3}_{\text{4 times}}$$

II a) on a calculator

There should be a key marked x^y or y^x. This is the key used to raise a number to a power.

For instance, 3^4 can be calculated by pressing the following key sequence:

- ➔ 3
- ➔ x^y
- ➔ 4

The answer you should get is 81.

Exercise 1.

For the following exercises, try to use first a pen and paper, and then validate it with the calculator.

1.1 7^2	1.2 3^3	1.3 2^4	1.4 4^3
1.5 1^3	1.6 10^3	1.7 3^4	1.8 0^3
1.9 2^6	1.10 4^4	1.11 6^2	1.12 5^3
1.13 2^7	1.14 6^3	1.15 2^8	1.16 7^3
1.17 10^4	1.18 5^4	1.19 4^5	1.20 2^9
1.21 0^4	1.22 9^2	1.23 6^4	1.24 8^3
1.25 8^2	1.26 3^5	1.27 5^5	1.28 9^3
1.29 1^{20}	1.30 3^6	1.31 12^2	1.32 12^3
1.33 2^{10}	1.34 4^6	1.35 30^2	1.36 8^4
1.37 2^{11}	1.38 20^3	1.39 2^{12}	

Exercise 2.
Rewrite the following replacing a or n with a positive whole number

2.1 $32 = 2^n$	2.2 $25 = a^2$	2.3 $1 = a^3$	2.4 $27 = 3^n$
2.5 $16 = a^4$	2.6 $125 = a^3$	2.7 $64 = 8^n$	2.8 $64 = 4^n$
2.9 $64 = 2^n$	2.10 $81 = 9^n$	2.11 $81 = 3^n$	2.12 $0 = a^{10}$
2.13 $512 = a^9$	2.14 $216 = 6^n$	2.15 $1000 = 10^n$	2.16 $1296 = a^4$
2.17 $625 = 5^n$	2.18 $1024 = a^{10}$	2.19 $729 = 9^n$	2.20 $1 = a^{15}$
2.21 $729 = 9^n$	2.22 $1024 = 4^n$	2.23 $343 = 7^n$	2.24 $243 = a^5$
2.25 $256 = a^8$	2.26 $256 = a^4$	2.27 $400 = a^2$	2.28 $2048 = 2^n$

II b) Square Roots

Def.

SQUARE ROOT AND $\sqrt{\ }$: A square root $\sqrt{x}$ of a number x is a value that can be multiplied by itself to give the original number $\sqrt{x} \times \sqrt{x} = x$.
Another notation for $\sqrt{x}$ is $x^{\frac{1}{2}}$; in other words, the square root of x is x to the power of a half.

For instance, the square root of 16 is 4, as $4 \times 4 = 16$.
But $(-4) \times (-4) = 16$! It means that -4 is also a square root of 16.
Computing the square root is usually a hard task. This is why it is done by using a calculator. The square root is found by using the $\sqrt{\ }$ key. When we use the $\sqrt{\ }$ symbol, the calculator refer to a *positive* number. Hence, $\sqrt{16} = 4$.
It is rarely an easy number to find: $\sqrt{20} = 4.472$ (to 3 decimal places).

Tip

$\sqrt{-4}$ does not exist for you calculator: you will get *error*.
You can not calculate the square root of a negative number. There is not a number that you can multiply by itself to get -4, both 2×2 and $-2 \times -2 = +4$.

Tip

Every time you find the square root of a number there are always two answers (although one is sometimes over looked). $\sqrt{16} = 4$, but we also have $-4 \times -4 = 16$. Every square root has a positive and a negative answer but calculators always give only the positive answer. Strictly speaking $\sqrt{4}$ is 2 and -2 which is sometimes written ± 2 (plus or minus 2).

There are a number of different ways that this can be calculated on your calculator.

- ➔ use the $\sqrt{\ }$ key
- ➔ use the $x^{\frac{1}{y}}$ key. For instance, for $\sqrt{16}$, you press 16, $x^{\frac{1}{y}}$, 2 and =. The 2 comes from $\sqrt{n} = n^{\frac{1}{2}}$
  note that with some calculators, the order might be reversed, i.e. the 2 might be before the $x^{\frac{1}{y}}$
- ➔ use the x^y key, and use the fraction key (ab/c) for the $\frac{1}{2}$. For instance, for $\sqrt{16}$, you press 16, x^y, 1, ab/c, 2 and =
- ➔ use the x^y key, and put the half into brackets. For instance, for $\sqrt{16}$, you press 16, x^y, $(1 \div 2)$ and =

Exercise 3.
Evaluate the following, give your answer to 3 significant figures where appropriate.

3.1 $\sqrt{4}$	3.2 $\sqrt{9}$	3.3 $\sqrt{20}$	3.4 $\sqrt{100}$	3.5 $\sqrt{1000}$
3.6 $\sqrt{64}$	3.7 $\sqrt{25}$	3.8 $\sqrt{10}$	3.9 $\sqrt{2}$	3.10 $\sqrt{1}$
3.11 $\sqrt{0}$	3.12 $\sqrt{-1}$	3.13 $\sqrt{5}$	3.14 $\sqrt{50}$	3.15 $\sqrt{60}$

Going further

You do not have a calculator ? No big deal.
You can have a decent approximation of a square root by following this algorithm.

1. start with a guess
   let's guess 4 is the square root of 20
2. divide by the guess
   $20/4 = 5$
3. add to the guess
   $4 + 5 = 9$
4. then divide the result by two
   $9/2 = 4.5$
5. Use it as the new guess, and start at 2)

For instance, calculating $\sqrt{20} = 4.472$:

- ➔ first guess is 4, which leads to 4.5
- ➔ The guess is 4.5, which leads to 4.722
- ➔ the guess is 4.722, which leads to 4.479
  It starts to be really close !
- ➔ the guess is 4.492, which leads to 4.472
  As close as precision used !

II c) More Roots

The cube root of x is x to the power of one third and the fourth root of x is x raised to the power of one quarter and so on.

Def.

ROOTS $\sqrt[n]{x}$: Roots can be generalized. The nth root of x, noted $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$, is a number that is equal to x when multiplied n times with itself:

$$\underbrace{\sqrt[n]{x} \times \sqrt[n]{x} \times \ldots \times \sqrt[n]{x}}_{\text{n times}} = x$$

For instance, $\sqrt[3]{8} = 2$, as $2 \times 2 \times 2 = 8$.

As seen in Sec. II b), square roots can be found using the square root key. Most calculators have a cube root key $\sqrt[3]{}$, but higher roots must be found using one of the methods described above:

- ➔ use the $x^{\frac{1}{y}}$ key. For instance, for $\sqrt[4]{16}$, you press 16, $x^{\frac{1}{y}}$, 4 and =. The n comes from $\sqrt[n]{x} = x^{\frac{1}{n}}$
  note that with some calculators, the order might be reversed, i.e. the n might be before the $x^{\frac{1}{y}}$

- ➔ use the x^y key, and use the fraction key (ab/c) for the $\frac{1}{n}$. For instance, for $\sqrt[4]{16}$, you press 16, x^y, 1, ab/c, 4 and =
- ➔ use the x^y key, and put the half into brackets. For instance, for $\sqrt[4]{16}$, you press 16, x^y, $(1 \div 4)$ and =

For instance, you should find that, up to the 5 significant figures:

- ➔ $\sqrt[3]{16} = 16^{\frac{1}{3}} = 2.5198$.
- ➔ $\sqrt[5]{20} = 20^{\frac{1}{5}} = 1.8206$.

Exercise 4.
Evaluate the following:

4.1	$\sqrt[3]{8}$	4.2	$\sqrt{9}$	4.3	$\sqrt[4]{16}$	4.4	$\sqrt{1}$	4.5	$\sqrt[3]{27}$
4.6	$\sqrt{36}$	4.7	$\sqrt{0}$	4.8	$\sqrt[6]{64}$	4.9	$\sqrt{81}$	4.10	$\sqrt[4]{81}$
4.11	$\sqrt[7]{128}$	4.12	$\sqrt[3]{1}$	4.13	$\sqrt[4]{216}$	4.14	$\sqrt[6]{729}$	4.15	$\sqrt[n]{1}$
4.16	$\sqrt[4]{1296}$	4.17	$\sqrt[3]{1728}$	4.18	$\sqrt[11]{2048}$	4.19	$\sqrt[n]{0}$	4.20	$\sqrt[3]{8000}$

Exercise 5.
Rewrite the following replacing a or n with a positive integer

5.1	$8 = \sqrt{a}$	5.2	$4 = \sqrt[n]{64}$	5.3	$2 = \sqrt[5]{a}$	5.4	$7 = \sqrt[n]{343}$
5.5	$4 = \sqrt[n]{256}$	5.6	$3 = \sqrt[5]{a}$	5.7	$1 = \sqrt[7]{a}$	5.8	$4 = \sqrt[n]{1024}$
5.9	$11 = \sqrt{a}$	5.10	$0 = \sqrt[5]{a}$	5.11	$100 = \sqrt{a}$	5.12	$27 = \sqrt{a}$
5.13	$36 = \sqrt{a}$	5.14	$8 = \sqrt[n]{512}$	5.15	$1 = \sqrt[50]{a}$	5.16	$9 = \sqrt[n]{729}$
5.17	$32 = \sqrt{a}$	5.18	$0 = \sqrt[50]{a}$	5.19	$8 = \sqrt[n]{4096}$	5.20	$16 = \sqrt[3]{a}$

III Laws of Indices

To manipulate expressions involving indices, we use rules known as the laws of indices.

Tip

The laws should be used precisely as they are given. Do not be tempted to make up variations of your own!

Def.

ZEROTH RULE : Anything to the power of 0 is equal to one.

$$a^0 = 1$$

Going further

Anything to the power of 0 is one. But also, zero to the power of anything is zero. ($0^m = 0$).
But what happens exactly when we try to have 0^0 ?
Let's use a calculator for looking at how a^b behave when we have a and b are going close to zero. Let's start with $a = 1$, $b = 1$, and let's decrease a and b in the next table.

a	b	a^b
1.0	1.0	1.0
0.9	0.9	0.90953
0.8	0.8	0.83651
0.7	0.7	0.77905
0.6	0.6	0.73602
0.5	0.5	0.70710
0.4	0.4	0.69314

It seems that it is going to zero. So, is $0^0 = 0$?
Let's continue closer to zero in the next table.

a	b	a^b
0.1	0.1	0.79432
0.01	0.01	0.95499
0.001	0.001	0.99311
0.0001	0.0001	0.99907

It now seems that it is going to 1! So, finally, do we have $0^0 = 1$? But, how is something times zero not equal to zero ?!
The statement 0^0 is ambiguous, and has actually been long debated in mathematics. This is mostly a matter of definition. It is considered to be an "indeterminate form,", but we will admit that $0^0 = 1$, because:

- ➔ it seems to be the case in the previous table
- ➔ it will be very useful in many cases

Prop.

PRODUCT RULE OF INDICES : When expressions with the same base are multiplied, the indices are added.

$$a^m \times a^n = a^{m+n}$$

It is easy to understand why:

$$\begin{aligned} a^{m+n} &= \underbrace{a \times a \times \ldots \times a}_{\text{m+n times}} \\ &= \underbrace{a \times a \times \ldots \times a}_{\text{m times}} \times \underbrace{a \times a \times \ldots \times a}_{\text{n times}} \end{aligned}$$

or:

- ➔ $a^m = \underbrace{a \times a \times \ldots \times a}_{\text{m times}}$
- ➔ $a^n = \underbrace{a \times a \times \ldots \times a}_{\text{n times}}$

Hence, $a^{m+n} = a^m \times a^n$.
Et voilà !

For instance

$$10^3 \times 10^4 = (10 \times 10 \times 10) \times (10 \times 10 \times 10 \times 10) = 10^7$$

Prop.

NEGATIVE EXPONENT RULE : if you move a term from the denominator to the numerator you must change the sign of the power.

$$\frac{1}{a^m} = a^{-m}$$

Let's chose n so $a^n = 1/a^m$. We have

$$\begin{aligned} a^m \frac{1}{a^m} &= \frac{a^m}{a^m} \\ &= a^0 \\ &= a^m \times a^n \\ &= a^{m+n} \end{aligned}$$

It means that $a^{m+n} = a^0$ and hence $m + n = 0$. Consequently, $n = -m$ and

$$a^n = a^{-m} = \frac{1}{a^m}$$

Et voilà !

For instance:

$$\frac{1}{10^3} = 10^{-3}$$

The same applies if you move from the numerator to the denominator:

$$10^2 = \frac{1}{10^{-2}}$$

Prop.

QUOTIENT RULE OF INDICES : When expressions with the same base are divided, the indices are substracted.

$$a^m/a^n = a^{m-n}$$

It is easy to understand why. a^m/a^n is:

$$\frac{a^m}{a^n} = \frac{\overbrace{a \times a \times a \times \ldots \times a}^{\text{m times}}}{\underbrace{a \times a \times \ldots \times a}_{\text{n times}}}$$

Let' s break up the demonstration in three cases:

1. Let' s suppose that $m > n$, then it means that we can cancel out some terms on the denominator:

$$\frac{a^m}{a^n} = \frac{a^m}{a^n} = \frac{\overbrace{\underbrace{\not{a} \times \not{a} \times \ldots \times \not{a}}_{\text{n times}} \ldots a \times a}^{\text{m times}}}{\not{a} \times \not{a} \times \ldots \times \not{a}}$$

Which leads to:

$$\frac{a^m}{a^n} = \underbrace{a \times a \times \ldots \times a}_{\text{m times with n cancelled, hence m-n times !}}$$

2. And what if $n > m$? Then it means that we can cancel out some terms on the numerator:

$$\frac{a^m}{a^n} = \frac{a^m}{a^n} = \frac{\not{a} \times \not{a} \times \ldots \times \not{a}}{\underbrace{\overbrace{\not{a} \times \not{a} \times \ldots \times \not{a}}^{\text{n times}} \ldots a \times a}_{\text{m times}}}$$

Which leads to:

$$\frac{a^m}{a^n} = \frac{1}{\underbrace{a \times a \times \ldots \times a}_{\text{n times with m cancelled, hence n-m times !}}}$$

or, $1/b^p = b^{-p}$, hence the result.

3. if $m = n$, then $a^m/a^n = 1$ which is a^0, and hence a^{m-n}.

Et voilà !

For instance

$$\begin{aligned} \frac{10^4}{10^3} &= \frac{10 \times 10 \times 10 \times 10}{10 \times 10 \times 10} \\ &= \frac{\not{10} \times \not{10} \times \not{10} \times 10}{\not{10} \times \not{10} \times \not{10}} \\ &= 10 \end{aligned}$$

Prop.

POWER LAW OF INDICES : If you raise one power to another power you multiply the powers together.

$$(a^m)^n = a^{m\times n}$$

$$\begin{aligned}(a^m)^n &= \overbrace{\underbrace{a\times a\times\ldots\times a}_{\text{m times}}\times\ldots\times\underbrace{a\times a\times\ldots\times a}_{\text{m times}}}^{\text{n times}}\\ &= \underbrace{a\times a\times\ldots\times a}_{m\times n\text{times}}\end{aligned}$$

Et voilà !

Prop.

FRACTIONAL RULE OF INDICES : The denominator of the fraction is the root of the number or letter, and the numerator of the fraction is the power to raise the answer to:

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

Let's say that p is such $\sqrt[n]{a^m} = a^p$.
We have, when raising to n:

$$\begin{aligned}(\sqrt[n]{a^m})^n &= (a^p)^n\\ &= a^{pn}\end{aligned}$$

Or, $(\sqrt[n]{a^m})^n = a^m$. It means that $pn = m$, and hence that $p = \dfrac{m}{n}$
Et voilà !

Let's look at $\sqrt[3]{27^2}$.
This can be, practically, calculated a number of ways:

- ➔ 27 raised to the power of two thirds using a variety of methods as in Section II a).
- ➔ 27 squared then cube rooted
- ➔ 27 cube rooted then squared $\sqrt[3]{27^2} = 27^{2/3} = (27^{1/3})^2 = 3^2 = 9$

Main Example

Let's simplify

$$b^3 \times a^4 \frac{b^3}{a^4} \times (a^2 b)^{2\div 3}$$

The first thing to do is to remove brackets:

$$\begin{aligned} (a^2 b)^{2\div 3} &= (a^2 b)^{\frac{2}{3}} \\ &= a^{2\times\frac{2}{3}} \times b^{1\times\frac{2}{3}} && \text{Using the power law} \\ &= a^{\frac{4}{3}} \times b^{\frac{2}{3}} \end{aligned}$$

Then, we can regroup terms:

$$\begin{aligned} b^3 \times \cancel{a^4}\frac{b^3}{\cancel{a^4}} \times a^{\frac{4}{3}} \times b^{\frac{2}{3}} &= 4b^3 \times b^3 \times b^{\frac{2}{3}} \times a^{\frac{4}{3}} && \text{regrouping by terms} \\ &= 4b^{3+3+\frac{2}{3}} \times a^{\frac{4}{3}} && \text{simplifying using product rule} \\ &= 4b^{\frac{20}{3}} \times a^{\frac{4}{3}} \end{aligned}$$

Exercise 6.

Simplify the following

6.1 $a^4 \times a^5$	6.2 $b^3 \times b^6$	6.3 $x \times x^7$	6.4 $5^2 \times 5^3$
6.5 $q^6 \times q^6$	6.6 $6^7 \times 6^8$	6.7 $10^x \times 10^x$	6.8 $a^3 \times a^4 \times a^5$
6.9 $p^2 \div p^2$	6.10 $3^7 \div 3^3$	6.11 $r^{10} \div r^4$	6.12 $6^8 \div 6^7$
6.13 $b^{20} \div b^{10}$	6.14 $k^n \div k^n$	6.15 $(p^2)^2$	6.16 $(q^4)^3$
6.17 $(\mathrm{M}^2)^7$	6.18 $(x^7)^3$	6.19 $(w^4)^n$	6.20 $(k^6)^4$
6.21 $a^7 \times a^{-3}$	6.22 $x^{-2} \times x^{-4}$	6.23 $10^{-4} \times 10$	6.24 $n^0 \times n^7$
6.25 $r^{0.5} \times r^{0.5}$	6.26 $a^5 \times a^{-5}$	6.27 $p^{3.5} \times p^{2.5}$	6.28 $c^{1-x} \times c^{1-x}$
6.29 $3^{-2} \div 3^2$	6.30 $w \div w^{0.5}$	6.31 $x^{-2} \div x^{-2}$	6.32 $f^0 \div f^4$
6.33 $10^{-1} \div 10^3$	6.34 $a^{-x} \div a^x$	6.35 $(x^{-2})^{-5}$	6.36 $(p^{0.5})^2$
6.37 $(10^{-1})^5$	6.38 $(r^{0.5})^3$	6.39 $(v^0)^8$	6.40 $w^{(1-x)(1+x)}$

Exercise 7.
Simplify the following

7.1 $4^{1/2}$	7.2 $9^{1/2}$	7.3 $27^{1/3}$	7.4 $1^{1/2}$
7.5 $243^{1/5}$	7.6 $0^{1/2}$	7.7 $64^{1/2}$	7.8 $64^{1/3}$
7.9 $64^{1/6}$	7.10 $128^{1/7}$	7.11 $512^{1/3}$	7.12 $4^{3/2}$
7.13 $32^{2/5}$	7.14 $8^{2/3}$	7.15 $27^{4/3}$	7.16 $1^{3/7}$
7.17 $128^{2/7}$	7.18 $64^{2/3}$	7.19 $9^{5/2}$	7.20 $4^{7/2}$
7.21 $81^{3/4}$	7.22 $8^{4/3}$	7.23 $256^{3/8}$	7.24 $1024^{0.2}$
7.25 $243^{0.6}$	7.26 $243^{4/5}$	7.27 $16^{7/4}$	7.28 $4096^{5/12}$
7.29 $\left(\frac{1}{2}\right)^{-3}$	7.30 $\frac{1}{2^{-3}}$	7.31 $\frac{1}{7^{-2}}$	7.32 $\left(\frac{1}{2}\right)^{-4}$
7.33 $(0.125)^{-}1$	7.34 7^0	7.35 $\frac{1}{3^{-3}}$	7.36 $\left(\frac{1}{2}\right)^{-5}$
7.37 $\frac{1}{14^{-1}}$	7.38 $\frac{1}{49^{-1/2}}$	7.39 $\left(\frac{1}{3}\right)^{-1}$	7.40 $\left(\frac{1}{27}\right)^{-1/3}$

IV Logarithm

Logarithms (and exponential) have been developed and used by/for accountants, due to their nice properties. Indeed, it is easy to add and substract quantities, but it is really hard to multiply or divide quantities. Think of the difference between $571 + 483$ vs 571×483.

Going further

The method of logarithms was publicly propounded by John Napier in 1614, in a book titled *Mirifici Logarithmorum Canonis Descriptio* (Description of the Wonderful Rule of Logarithms). The name Logarithm is the combination of two Greek roots, Logos (reason or ratio) + artihmus (number). The ratio refers to the original method of constructing logarithms by geometric sequences. The name was introduced by John Napier.
Pierre-Simon Laplace called logarithms an " admirable artifice which, by reducing to a few days the labour of many months, doubles the life of the astronomer, and spares him the errors and disgust inseparable from long calculations."
They were the main way of achieving hard computations, before the introduction of computers and calculators.

Whilst logarithms are no longer used to multiply and divide, their use is not obsolete, they (and their rules) are used to solve equations where the power is the unknown, e.g. $32 = 5^x$.

It is also widely use in data science.

Def.

LOGARITHM: A logarithm is defined as the inverse of an index:

$$b^x = y \rightsquigarrow \log_b(y) = x$$

It holds as long as:

- ➔ $y > 0$
- ➔ $b \neq 1$

Going further

We are going to use the abbreviation "log" and "Log" a lot in the following. But where does it come from ?
According to Gordon Fisher, Log. (with a period and a capital "L") was first introduced by Johannes Kepler (1571-1630) in 1624 in Chilias logarithmorum.
The abbreviation log. (without the capital letter, still with a period) was used a few years later, by Bonaventura Cavalieri (1598-1647) in Directorium generale Vranometricum in 1632 (Cajori vol. 2, page 106).
Finally, the abbreviation that we use, log (without a period, lower case "l") appears in the 1647 edition of Clavis mathematicae by William Oughtred.

Def.

BASE: The log of a number to a base is the power to which the base must be raised to get that number"
It means that the number b in $\log_b$ is named the base of the logarithm.

For instance, if $10^x = 4$, then $x = \log_{10} 4$ and $\log_{10}$ is a logarithm in base 10.
There are two commonly used logs,

- ➔ logs to base 10 ($\log_{10}$)
- ➔ logs to base e (log_e)
  e is a special number ($e \approx 2.718281828$).

Logarithm to base 10 is the "Log" key on your calculator. Logarithm to base e is the "ln" key. The "ln" comes from the name of logarithm to base e: it is called Natural Logarithm (or Naperian logarithm), after Napier the mathematician who discovered logarithms.

Going further

Can we show that the function $y = b^x$ has a solution for any $y > 0$ and $b \neq 1$?
If it does not, then our definition is wrong ! That would be embarrassing...
The answer is of course yes but not that easy. Let's illustrate it in a simple case: $b = 2$. Roughly speaking, we know that

- ➔ when x is really small (for instance, $x = -10$), 2^x is really small: $2^-10 \approx 0.001$.
- ➔ when x is really large (for instance, $x = 20$), 2^x is really large: $2^20 = 1048576$.
- ➔ 2^x is always increasing when x increase: $2^5 > 2^4 > 2^3 \dots$.

That means that if you pick a number y between 0.001 and 1048576, you can find an x between -10 and 20 such as $2^x = y$. Because 2^x is increasing, you know that this number is unique !
The real proof is in the same lines, and use the so called *intermediate value theorem.*

IV a) Laws of Logs

In the following, we consider that the logs are in the same base (i.e., base e).

Def.

FUNDAMENTAL LAW OF LOGS (LAW 1): The logarithm of a product is the sum of the logarithms of the factors:

$$\log A + \log B = \log AB$$

This property is fundamental and is the main reason of the invention of logarithms.

Prop.

POWER: LAW 2: The logarithm of a factor A to the power n is equal the n the logarithm of A:

$$\log A^n = n \log A$$

This property is derived from the law 1:

$$\begin{aligned} \log A^n &= \log \underbrace{A \times A \times \ldots \times A}_{\text{n times}} \\ &= \underbrace{\log A + \log A + \ldots + \log A}_{\text{n times}} \quad \textit{law1} \\ &= n \log A \end{aligned}$$

Et voilà !

Prop.

DIVISION AND SUBTRACTION: LAW 3: The logarithm of a fraction is the difference of the logarithms of the factors:

$$\log A - \log B = \log \frac{A}{B}$$

This property is derived from the law 1 and law 2:

$$\begin{aligned} \log \frac{A}{B} &= \log AB^{-1} \\ &= \log A + \log B^{-1} & \textit{law2} \\ &= \log A - \log B & \textit{law1} \end{aligned}$$

Et voilà !

This law also shows that $\log 1 = 0$!

Main Example

A few examples:

- ➔ $\log 4 + \log 7 = \log(4 \times 7)$, using law 1
- ➔ $2 \log 3 = \log\left(3^2\right) = \log 9$, using law 2
- ➔ $\log 8 - \log 2 = \log\left(\frac{8}{2}\right) = \log 4$, using law 3

Exercise 8.

Simplify the following, expressing each as a single log term

all answers are integers

8.1 $\log 6 + \log 5$

8.2 $\log 2 + \log 10$

8.3 $\log 10 - \log 2$

8.4 $\log 3 + \log 4 + \log 5$

8.5 $\log 12 + \log 2$

8.6 $\log 8 - \log 4$

8.7 $\log 5 + \log 5$

8.8 $\log 6 + \log 1$

8.9 $\log 12 - \log 3 - \log 4$

8.10 $\log 5 - \log 5$

8.11 $3 \log 2$

8.12 $4 \log 4$

8.13 $\frac{1}{2} \log 9$

8.14 $2 \log 6 - \log 12$

8.15 $3 \log 3 - \log 6$

8.16 $2 \log 4 - 3 \log 2$

8.17 $2 \log 4 + \log 2$

8.18 $\frac{1}{4} \log 16$

8.19 $3 \log 1$

8.20 $4 \log 2 - 2 \log 4$

8.21 $3 \log 5 - \frac{1}{2} \log 25$

8.22 $\frac{1}{4} \log 256 - \frac{1}{2} \log 4$

IV b) More about base 10 and base e

IV b) i Logs to base 10

Let's start with an illustration: $\log_{10} 100 = 2$, because 2 is the power to wich 100 has to be raised to get 100: $10^2 = 100 \rightsquigarrow \log_{10} 100 = 2$.

The notation in base 10 is very important as the core of scientific notations. They represent very easily big numbers and small numbers:

- ➔ instead of 2500000000000000, we can write 2.5×10^{15}.
- ➔ the number of atoms in 12 grams of carbone is roughly 602200000000000000000000, or 6.022×10^{23}.
- ➔ instead of 0.00000000000000012, we can write 1.2×10^{-16}.
- ➔ the size of the HIV virus is 0.000000120m, or 120×10^{-9}m.

Going further

Logarithm to base 10 are from time to time called Briggsian logarithm.

Henry Briggs was a British mathematician from the XVIIth century. He calculated tables in base 10 in 1624, in the book *Arithmetica Logarithmetica.* Tables remained used until the 70's. Because logarithms were so useful, tables of base-10 logarithms were given in appendices of many textbooks !

IV b) ii Logs to base e

Denoted by ln, logs to base e have a special meaning in mathematics. It will become clearer when the gradient of a function is examined in calculus, but it is called "natural" because it is linked to simple functions. The e comes from the infamous exponential function (as in exponential growth).

Going further

e is named "Euler's number".
It is:

$$e = 2.71828\ldots$$

Leonhard Euler introduced the letter e as the base for natural logarithms, writing in a letter to Christian Goldbach of 25 November 1731.
But why "natural logarithm" ?
The natural logarithm can be defined for any positive real number a as the area under the curve $y = 1/x$, between 1 and a. The "simplicity" of this definition has lead Gregoire de Saint-Vincent's to name it "natural".

Both types of logs can be used to solve equations where the power is unknown as the following example shows.

Main Example

Let's try to solve, for x,

$$15^x = 40$$

The first step is to "take down" the index.

$15^x = 40$ using the log $x \log_{10} 15 = \log_{10} 40$

For that, we do:

1. We have $15^x = 40$
2. We take the log, for instance to base 10:
   we now have $\log_{10} 15^x = \log_{10} 40$
3. We use the law 2:
   we now have $x \log_{10} 15 = \log_{10} 40$
4. we can now identify x: $x = \dfrac{\log_{10} 40}{\log_{10} 15} = 1.26219$

Let's verify the solution:

$$15^{1.26219} = 39.9999797$$

We have rounded x to 5 decimals, and hence the slight error.
We can do the same with the base e:

1. We have $15^x = 40$
2. We take the log e:
   we now have $\log_e 15^x = \log_e 40$
3. We use the law 2:
   we now have $x \log_e 15 = \log_e 40$
4. we can now identify x: $x = \dfrac{\log_e 40}{\log_e 15} = 1.26219$

Exercise 9.
Solve the following for either n, x or a. When relevant, use 3 decimals.

9.1 $\log_2 8 = n$	9.2 $\log_3 9 = n$	9.3 $\log_4 64 = n$	9.4 $\log_6 6 = n$
9.5 $\log_2 32 = n$	9.6 $\log_6 36 = n$	9.7 $\log_7 49 = n$	9.8 $\log_5 125 = n$
9.9 $\log_4 4 = n$	9.10 $\log_3 81 = n$	9.11 $\log_7 n = 1$	9.12 $\log_2 x = 4$
9.13 $\log_5 x = 4$	9.14 $\log_3 x = 5$	9.15 $\log_3 7 = n$	9.16 $\log_5 14 = n$
9.17 $\log_2 9 = n$	9.18 $n \log_2 8 = 6$	9.19 $\log_3 1 = n$	9.20 $\log_a 7 = 1$

9.21 $\log_a 7 = 0$	9.22 $2.7 = 10^x$	9.23 $3.5 = 5^x$	9.24 $4.1 = 2^x$
9.25 $25 = 3^x$	9.26 $256 = 7^x$	9.27 $1 = 3.2^x$	9.28 $13 = 5.6^{-x}$
9.29 $21 = \frac{1}{2^x}$	9.30 $21 = 2^x$	9.31 $34 = 3.4^x$	9.32 $5 = 2^x$

V Solutions to exercises

Solution 1.

1.1 $7^2 = 49$	1.2 $3^3 = 27$	1.3 $2^4 = 16$	1.4 $4^3 = 64$
1.5 $1^3 = 1$	1.6 $10^3 = 1000$	1.7 $3^4 = 81$	1.8 $0^3 = 0$
1.9 $2^6 = 64$	1.10 $4^4 = 256$	1.11 $6^2 = 36$	1.12 $5^3 = 125$
1.13 $2^7 = 128$	1.14 $6^3 = 216$	1.15 $2^8 = 256$	1.16 $7^3 = 343$
1.17 $10^4 = 10000$	1.18 $5^4 = 625$	1.19 $4^5 = 1024$	1.20 $2^9 = 512$
1.21 $0^4 = 0$	1.22 $9^2 = 81$	1.23 $6^4 = 1296$	1.24 $8^3 = 512$
1.25 $8^2 = 64$	1.26 $3^5 = 243$	1.27 $5^5 = 3125$	1.28 $9^3 = 729$
1.29 1^{20}	1.30 $3^6 = 729$	1.31 $12^2 = 144$	1.32 $12^3 = 1728$
1.33 $2^{10} = 1024$	1.34 $4^6 = 4096$	1.35 $30^2 = 900$	1.36 $8^4 = 4096$
1.37 $2^{11} = 2048$	1.38 $20^3 = 8000$	1.39 $2^{12} = 4096$	

Solution 2.

2.1 $32 = 2^5$	2.2 $25 = 5^2$	2.3 $1 = 1^3$	2.4 $27 = 3^3$
2.5 $16 = 2^4$	2.6 $125 = 5^3$	2.7 $64 = 8^2$	2.8 $64 = 4^3$
2.9 $64 = 2^6$	2.10 $81 = 9^2$	2.11 $81 = 3^4$	2.12 $0 = 0^{10}$
2.13 $512 = 2^9$	2.14 $216 = 6^3$	2.15 $1000 = 10^3$	2.16 $1296 = 6^4$
2.17 $625 = 5^4$	2.18 $1024 = 2^{10}$	2.19 $729 = 9^3$	2.20 $1 = 1^{15}$
2.21 $729 = 9^3$	2.22 $1024 = 4^5$	2.23 $343 = 7^3$	2.24 $243 = 3^5$
2.25 $256 = 2^8$	2.26 $256 = 4^4$	2.27 $400 = 20^2$	2.28 $2048 = 2^{11}$

Solution 3.

3.1 $\sqrt{4} = \pm 2$	3.2 $\sqrt{9} = \pm 3$	3.3 $\sqrt{20} = \pm 4.47$	3.4 $\sqrt{100} = \pm 10$
3.5 $\sqrt{1000} = \pm 31.6$	3.6 $\sqrt{64} = \pm 8$	3.7 $\sqrt{25} = \pm 5$	3.8 $\sqrt{10} = \pm 3.16$
3.9 $\sqrt{2} = \pm 1.41$	3.10 $\sqrt{1} = \pm 1$	3.11 $\sqrt{0} = \pm 0$	3.12 $\sqrt{-1}$ is not defined
3.13 $\sqrt{5} = \pm 2.24$	3.14 $\sqrt{50} = \pm 7.07$	3.15 $\sqrt{60} = \pm 7.75$	

Solution 4.

4.1	$\sqrt[3]{8}=2$	4.2	$\sqrt{9}=3$	4.3	$\sqrt[4]{16}=2$	4.4	$\sqrt{1}=1$	4.5	$\sqrt[3]{27}=3$
4.6	$\sqrt{36}=6$	4.7	$\sqrt{0}=0$	4.8	$\sqrt[6]{64}=2$	4.9	$\sqrt{81}=9$	4.10	$\sqrt[4]{81}=3$
4.11	$\sqrt[7]{128}=2$	4.12	$\sqrt[3]{1}=1$	4.13	$\sqrt[3]{216}=6$	4.14	$\sqrt[6]{729}=2$	4.15	$\sqrt[n]{1}=1$
4.16	$\sqrt[4]{1296}=6$	4.17	$\sqrt[3]{1728}=12$	4.18	$\sqrt[11]{2048}=2$	4.19	$\sqrt[n]{0}=1$	4.20	$\sqrt[3]{8000}=20$

Solution 5.

5.1	$8=\sqrt{64}$	5.2	$4=\sqrt[3]{64}$	5.3	$2=\sqrt[5]{32}$	5.4	$7=\sqrt[3]{343}$
5.5	$4=\sqrt[4]{256}$	5.6	$3=\sqrt[5]{243}$	5.7	$1=\sqrt[7]{1}$	5.8	$4=\sqrt[5]{1024}$
5.9	$11=\sqrt{121}$	5.10	$0=\sqrt[5]{0}$	5.11	$100=\sqrt{10000}$	5.12	$27=\sqrt{729}$
5.13	$36=\sqrt{1296}$	5.14	$8=\sqrt[3]{512}$	5.15	$1=\sqrt[50]{1}$	5.16	$9=\sqrt[3]{729}$
5.17	$32=\sqrt{1024}$	5.18	$0=\sqrt[50]{0}$	5.19	$8=\sqrt[4]{4096}$	5.20	$16=\sqrt[3]{4096}$

Solution 6.

6.1	$a^4\times a^5=a^9$	6.2	$b^3\times b^6=b^9$	6.3	$x\times x^7=x^8$
6.4	$5^2\times 5^3=5^5=3125$	6.5	$q^6\times q^6=q^{12}$	6.6	$6^7\times 6^8=6^{15}=470184984576$
6.7	$10^x\times 10^x=10^{2x}$	6.8	$a^3\times a^4\times a^5=a^{12}$	6.9	$p^2\div p^2=p^0=1$
6.10	$3^7\div 3^3=3^4=81$	6.11	$r^{10}\div r^4=r^6$	6.12	$6^8\div 6^7=6^1=6$
6.13	$b^{20}\div b^{10}=b^{10}$	6.14	$k^n\div k^n=k^0=1$	6.15	$(p^2)^2=p^4$
6.16	$(q^4)^3=q^{12}$	6.17	$(\mathrm{M}^2)^7=\mathrm{M}^{14}$	6.18	$(x^7)^3=x^{21}$
6.19	$(w^4)^n=w^{4n}$	6.20	$(k^6)^4=k^{24}$	6.21	$a^7\times a^{-3}=a^4$
6.22	$x^{-2}\times x^{-4}=x^{-6}$	6.23	$10^{-4}\times 10=10^{-3}$	6.24	$n^0\times n^7=n^7$
6.25	$r^{0.5}\times r^{0.5}=r$	6.26	$a^5\times a^{-5}=1$	6.27	$p^{3.5}\times p^{2.5}=p$
6.28	$c^{1-x}\times c^{1-x}=c^{2(1-x)}$	6.29	$3^{-2}\div 3^2=3^{-4}$	6.30	$w\div w^{0.5}=w^{0.5}=\sqrt{w}$
6.31	$x^{-2}\div x^{-2}=1$	6.32	$f^0\div f^4=f^{-4}$	6.33	$10^{-1}\div 10^3=10^{-4}=0.0001$
6.34	$a^{-x}\div a^x=a^{-2x}$	6.35	$(x^{-2})^{-5}=x^10$	6.36	$(p^{0.5})^2=p$
6.37	$(10^{-1})^5=10^{-5}=0.00001$	6.38	$(r^{0.5})^3=r^{1.5}=r\sqrt{r}$	6.39	$(v^0)^8=1$
6.40	$w^{(1-x)(1+x)}=w^{1-x^2}$				

Solution 7.

7.1 $4^{1/2} = 2$

7.2 $9^{1/2} = 3$

7.3 $27^{1/3} = 3$

7.4 $1^{1/2} = 1$

7.5 $243^{1/5} = 3$

7.6 $0^{1/2} = 0$

7.7 $64^{1/2} = 8$

7.8 $64^{1/3} = 4$

7.9 $64^{1/6} = 4$

7.10 $128^{1/7} = 2$

7.11 $512^{1/3} = 2$

7.12 $4^{3/2} = 8$

7.13 $32^{2/5} = 4$

7.14 $8^{2/3} = 4$

7.15 $27^{4/3} = 81$

7.16 $1^{3/7} = 1$

7.17 $128^{2/7} = 4$

7.18 $64^{2/3} = 16$

7.19 $9^{5/2} = 243$

7.20 $4^{7/2} = 128$

7.21 $81^{3/4} = 27$

7.22 $8^{4/3} = 16$

7.23 $256^{3/8} = 8$

7.24 $1024^{0.2} = 4$

7.25 $243^{0.6} = 27$

7.26 $243^{4/5} = 81$

7.27 $16^{7/4} = 128$

7.28 $4096^{5/12} = 32$

7.29 $\left(\frac{1}{2}\right)^{-3} = 8$

7.30 $\frac{1}{2^{-3}} = 8$

7.31 $\frac{1}{7^{-2}} = 49$

7.32 $\left(\frac{1}{2}\right)^{-4} = 16$

7.33 $(0.125)^{-}1 = 8$

7.34 $7^0 = 1$

7.35 $\frac{1}{3^{-3}} = 27$

7.36 $\left(\frac{1}{2}\right)^{-5} = 32$

7.37 $\frac{1}{14^{-1}} = 14$

7.38 $\frac{1}{49^{-1/2}} = 7$

7.39 $\left(\frac{1}{3}\right)^{-1} = 3$

7.40 $\left(\frac{1}{27}\right)^{-1/3} = 3$

Solution 8.

8.1 $\log 6 + \log 5 = \log 30$

8.2 $\log 2 + \log 10 = \log 20$

8.3 $\log 10 - \log 2 = \log 5$

8.4 $\log 3 + \log 4 + \log 5 = \log 60$

8.5 $\log 12 + \log 2 = \log 24$

8.6 $\log 8 - \log 4 = \log 2$

8.7 $\log 5 + \log 5 = \log 25$

8.8 $\log 6 + \log 1 = \log 6$

8.9 $\log 12 - \log 3 - \log 4 = \log 1 = 0$

8.10 $\log 5 - \log 5 = \log 1 = 0$

8.11 $3\log 2 = \log 8$

8.12 $4\log 4 = \log 256$

8.13 $\frac{1}{2}\log 9 = \log 3$

8.14 $2\log 6 - \log 12 = \log 1 = 0$

8.15 $3\log 3 - \log 6 = \log \frac{9}{2}$, tricky, it is not an integer !

8.16 $2\log 4 - 3\log 2 = \log 2$

8.17 $2\log 4 + \log 2 = \log 32$

8.18 $\frac{1}{4}\log 16 = \log 2$

8.19 $3\log 1 = \log 1 = 0$

8.20 $4\log 2 - 2\log 4 = \log 1 = 0$

8.21 $3\log 5 - \frac{1}{2}\log 25 = \log 25$

8.22 $\frac{1}{4}\log 256 - \frac{1}{2}\log 4 = \log 2$

Solution 9.

9.1 $\log_2 8 = 3$

9.2 $\log_3 9 = 2$

9.3 $\log_4 64 = 3$

9.4 $\log_6 6 = 1$

9.5 $\log_2 32 = 5$

9.6 $\log_6 36 = 2$

9.7 $\log_7 49 = 2$

9.8 $\log_5 125 = 3$

9.9 $\log_4 4 = 1$

9.10 $\log_3 81 = 4$

9.11 $\log_7 7 = 1$

9.12 $\log_2 16 = 4$

9.13 $\log_5 625 = 4$

9.14 $\log_3 243 = 5$

9.15 $\log_3 7 = 1.771$

9.16 $\log_5 14 = 1.640$

9.17 $\log_2 9 = 3.170$

9.18 $n\log_2 8 = 6$

9.19 $\log_3 1 = 0$

9.20 $\log_7 7 = 1$

9.21 $\log_a 7 = 0$ is impossible

9.22 $2.7 = 10^{0.431}$

9.23 $3.5 = 5^{0.778}$

9.24 $4.1 = 2^{2.036}$

9.25 $25 = 3^{2.930}$

9.26 $256 = 7^{2.850}$

9.27 $1 = 3.2^{0}$

9.28 $13 = 5.6^{-(-1.489)} = 5.6^{1.489}$

9.29 $21 = \frac{1}{2^{-4.392}}$

9.30 $21 = 2^{4.392}$

9.31 $34 = 3.4^{2.882}$

9.32 $5 = 2^{2.322}$

Bibliography

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