

Mathematics for Engineers–ENG 3009, 2018-2019

Quadratic equations

FLORIMOND GUENIAT & VIJAY VENKATESH
WITH BRIAN SMITH



BIRMINGHAM CITY
University

In this chapter, we will look at quadratic equations. Quadratic equations arise often in

- mathematics, for instance when looking at arithmetic series
- physics, e.g. when looking at trajectories of mortars
- engineering, like in constructing parabolas for radars, TV and phones.

As always, please free to refer to the book [Croft and Davidson, 2016] for details.

Introduction

A quadratic equation is one that contains the variable squared, as opposed to a linear equation that only contains the variable to the power 1.

If

$$y = 3x + 4$$

is an example of a linear equation,

$$y = 2x^2 + 3x + 4$$

is an example of a quadratic equation.

Quadratic equations occur in many areas and there are a number of methods that can be used to solve them.

I Standard form

The name given to the *shape* of a quadratic equation is a parabola and this can be seen in a number of real life situations such as the shape that of a parabolic mirror, see Fig. 1.

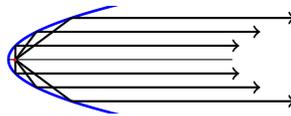


Figure 1: Sketch of a parabola

$$y = ax^2 + bx + c$$

is the standard form for a quadratic equation.

Tip

a , b and c can be any number, except that a cannot be 0, (if a is 0 then the equation is not a quadratic, it is linear).

The most simple quadratic equation, $y = x^2$, is shown below in Fig. 2(a) and the graph of $y = x^2$ is shown in Fig. 2(b).

There are a number of features about a quadratic that apply to all.

- They are a perfect parabola (hence the name)
- They are a symmetric about a vertical line,
 - $y = x^2$ this line is the y axis
 - $y = x^2 + 2x + 1$, this line is the line $y = -1$
- All can be derived from the graph of $y = x^2$ by various stretches and translations and eventually one flip around the x axis.

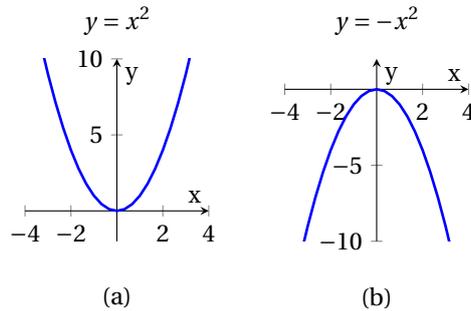


Figure 2: Plots of $y = x^2$ and $y = -x^2$

- All quadratics with a positive co-efficient of x^2 will have a minimum value like the one in Fig. 2(a).
- All quadratics with a negative co-efficient of x will have a maximum value like the one in Fig.2(b).

If we take the values of the quadratic when it crosses the x axis (i.e. the value $y = 0$) we are solving the quadratic equation $ax^2 + bx + c = 0$. This unit looks at various ways of doing this.

II Graphical method for solving $ax^2 + bx + c = 0$

To solve the equation $ax^2 + bx + c = 0$, we plot the graph of $y = ax^2 + bx + c$ and find where the curve crosses the x axis. First we must calculate y for various values of x . This might be best done in table form, or using a spreadsheet (like Excel or Open Office).

Main Example

For instance, let's consider

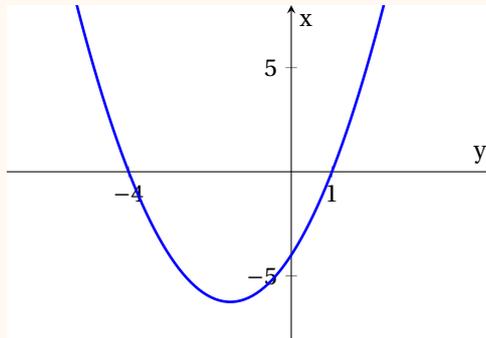
$$x^2 + 3x - 4 = 0$$

It means $a = 1$, $b = 3$ and $c = -4$.

We can construct the following table:

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
x^2	36	25	16	9	4	1	0	1	4	9	16	25	36
$3x$	-18	-15	-12	-9	-6	-3	0	3	6	9	12	15	18
-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
y	14	6	0	-4	-6	-6	-4	0	6	14	24	36	50

If we now plot these points on suitable axes, we get the following graph, and we see that $x = 4$ and 1 .



The two values of x , $x = 4$ and $x = 1$ are the only two values that will satisfy the equation $x^2 + 3x - 4 = 0$.

→ When $x = -4$: $x^2 + 3x - 4 = (-4)^2 + 3(-4) - 4 = 16 - 12 - 4 = 0$ as required

→ When $x = 1$: $x^2 + 3x - 4 = (1)^2 + 3(1) - 4 = 1 + 3 - 4 = 0$ as required

Exercise 1.

Solve the following quadratic equations graphically, by plotting a graph between $x = -4$ and $x = 5$

1.1 $y = x^2 - 2x - 7$

1.2 $y = x^2 - 2x + 1$

1.3 $y = -x^2 + 2x + 7$

III Factorization method

Sometimes, the expression $ax^2 + bx + c$ can be factorized, i.e., rewritten as a product of brackets.

For instance, $x^2 + 6x + 8$ can be factorized in $(x + 2)(x + 4)$.

$$x^2 + 6x + 8 = 0 \Rightarrow (x + 2)(x + 4) = 0$$

To check, we can remove the bracket.

$$\begin{aligned} (x + 2)(x + 4) &= x(x + 4) + 2(x + 4) && \text{split the 1st brackets} \\ &= x^2 + \underbrace{4x + 2x}_{\text{the } 6x} + \underbrace{2 \times 4}_{\text{the } 8} && \text{split the brackets} \end{aligned}$$

→ The 8 in $x^2 + 6x + 8$ corresponds to the 2×4 :

$$\begin{aligned} &2 \times 4 = 8 \\ &\underbrace{\hspace{1.5cm}} \\ (x + 2)(x + 4) &= x^2 + 6x + 8 \end{aligned}$$

Do we now have the solutions ? If $(x + 2)(x + 4) = 0$, then one of the brackets has to be 0:

- if $(x + 2) = 0$ then $x = -2$. Let's check:

$$x^2 + 6x + 8 = (-2)^2 + 6(-2) + 8 = 4 - 12 + 8 = 0$$

as required.

- if $(x + 4) = 0$ then $x = -4$. Let's check:

$$x^2 + 6x + 8 = (-4)^2 + 6(-4) + 8 = 16 - 24 + 8 = 0$$

as required.

So our two answers are -2 and -4 .

So if we are looking to factorize the brackets as $(x + e)(x + f)$, we want e and f so $e \times f = 8$, or, in the general case, c/a .

→ Similarly, the $6x$ comes from $2x + 4x = (2 + 4)x$. So if we are looking to factorize the brackets as $(x + e)(x + f)$, we want e and f so $e + f = 6$, or, in the general case, b/a .

We hence have an algorithm to factorize $ax^2 + bx + c = 0$ into a form $(x + e)(x + f) = 0$:

1. simplify by a: $ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
2. find e and f so:

$$\begin{cases} ef = \frac{c}{a} \\ e + f = \frac{b}{a} \end{cases}$$

3. conclude that $ax^2 + bx + c = a(x + e)(x + f) \Rightarrow (x + e)(x + f) = 0$

Tip

The factorization method works *well* only if e and f are easy numbers. If you can not find the solution right away, move to another method !

Let's solve by factorization the following quadratic equations:

$$x^2 + 7x + 12 = 0$$

We want to put it in the form of $(x + \underbrace{?}_e)(x + \underbrace{??}_f) = 0$, where:

$$\rightarrow ? + ?? = 7$$

$$\rightarrow ? \times ?? = 12$$

Note that here, $a = 1$, so $c/a = c$ and $b/a = b$.

There are three pairs of numbers that multiply together to give 12:

1. 1 and 12
2. 2 and 6
3. 3 and 4

Because we want them to sum up to 7:

1. $1 + 12 \neq 7$
2. $2 + 6 \neq 7$
3. $3 + 4 = 7$

It means that

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

We want to find the roots of the equation: $(x + 3)(x + 4) = 0$ means that either

$$\rightarrow x + 3 = 0 \Rightarrow x = -3$$

$$\rightarrow x + 4 = 0 \Rightarrow x = -4$$

You can verify that $(-3)^2 + 7(-3) + 12 = 0$ and $(-4)^2 + 7(-4) + 12 = 0$.

Let's solve by factorization the following quadratic equations:

$$2x^2 + 10x + 12 = 0$$

i.e., $a = 2, b = 10, c = 12$. We want to put it in the form of $(x + e)(x + f) = 0$, where:

$$\rightarrow e + f = b/a = 5$$

$$\rightarrow e \times f = c/a = 6$$

There are two pairs of numbers that multiply together to give 6:

1. 1 and 6
2. 2 and 3

Because we want them to sum up to $b/a = 5$:

1. $1 + 6 \neq 5$
2. $2 + 3 = 5$

It means that

$$2x^2 + 10x + 12 = 2(x + 2)(x + 3)$$

We want to find the roots of the equation: $(x + 2)(x + 3) = 0$ means that either

$$\rightarrow x + 2 = 0 \Rightarrow x = -2$$

$$\rightarrow x + 3 = 0 \Rightarrow x = -3$$

You can verify that $2(-2)^2 + 10(-2) + 12 = 0$ and $2(-3)^2 + 10(-3) + 12 = 0$.

Let's solve by factorization the following quadratic equations:

$$\rightarrow x^2 + 4x - 21 = 0:$$

Note that here, $a = 1$, so $c/a = c$ and $b/a = b$.

We want to put it in the form of $(x + \underbrace{?}_e)(x + \underbrace{??}_f) = 0$, where:

- o $? + ?? = 4$
- o $? \times ?? = -21$

We have a negative number -21 , it means that one number must be *positive*, and the other number has to be *negative*. There are two pairs of numbers that multiply together to give 21, and hence 4 combinations that gives -21 :

- o $1 \times 21 = 21$
 1. 1 and -21
 2. -1 and 21
- o $3 \times 7 = 21$
 1. 3 and -7
 2. -3 and 7

Because we want them to sum up to 4:

1. $-1 + 21 \neq 4$
2. $1 + (-21) \neq 4$

3. $3 + (-7) \neq 4$

4. $-3 + 7 = 4$

It means that

$$x^2 + 4x - 21 = (x - 3)(x + 7)$$

We want to find the roots of the equation: $(x - 3)(x + 7) = 0$ means that either

◦ $x - 3 = 0 \Rightarrow x = 3$

◦ $x + 7 = 0 \Rightarrow x = -7$

You can verify that $(3)^2 + 4(3) - 21 = 0$ and $(-7)^2 + 4(-7) - 21 = 0$.

→ $x^2 - 5x - 24 = 0$:

Note that here, $a = 1$, so $c/a = c$ and $b/a = b$.

We want to put it in the form of $(x + e)(x + f) = 0$, where:

◦ $e + f = -5$

◦ $e \times f = -24$

We have a negative number -24 , it means that one number must be *positive*, and the other number has to be *negative*. e and f can not have the same sign.

There are four pairs of numbers that multiply together to give 24, and hence 8 combinations that gives -24 :

◦ $1 \times 24 = 24$

1. 1 and -24

2. -1 and 24

◦ $2 \times 12 = 24$

1. 2 and -12

2. -2 and 12

◦ $3 \times 8 = 24$

1. 3 and -8

2. -3 and 8

◦ $4 \times 6 = 24$

1. 4 and -6

2. -4 and 6

Because we want them to sum up to -5 :

1. $-1 + 24 \neq -5$

2. $1 - 24 \neq -5$

3. $-2 + 12 \neq -5$

4. $2 - 12 \neq -5$

5. $-3 + 8 \neq -5$

6. $3 - 8 = -5$

It means that

$$x^2 - 5x - 24 = (x + 3)(x - 8)$$

We want to find the roots of the equation: $(x + 3)(x - 8) = 0$ means that either

- $x + 3 = 0 \Rightarrow x = -3$
- $x - 8 = 0 \Rightarrow x = 8$

You can verify that $(-3)^2 - 5(-3) - 24 = 0$ and $(8)^2 - 5(8) - 24 = 0$.

→ $x^2 - 5x - 24 = 0$:

Note that here, $a = 1$, so $c/a = c$ and $b/a = b$.

We want to put it in the form of $(x + e)(x + f) = 0$, where:

- $e + f = -5$
- $e \times f = -24$

We have a negative number -24 , it means that one number must be *positive*, and the other number has to be *negative*. e and f can not have the same sign.

There are four pairs of numbers that multiply together to give 24, and hence 8 combinations that gives -24 :

- $1 \times 24 = 24$
 1. 1 and -24
 2. -1 and 24
- $2 \times 12 = 24$
 1. 2 and -12
 2. -2 and 12
- $3 \times 8 = 24$
 1. 3 and -8
 2. -3 and 8
- $4 \times 6 = 24$
 1. 4 and -6
 2. -4 and 6

Because we want them to sum up to -5 :

1. $-1 + 24 \neq -5$
2. $1 - 24 \neq -5$
3. $-2 + 12 \neq -5$
4. $2 - 12 \neq -5$
5. $-3 + 8 \neq -5$
6. $3 - 8 = -5$

It means that

$$x^2 - 5x - 24 = (x + 3)(x - 8)$$

We want to find the roots of the equation: $(x + 3)(x - 8) = 0$ means that either

- $x + 3 = 0 \Rightarrow x = -3$
- $x - 8 = 0 \Rightarrow x = 8$

You can verify that $(-3)^2 - 5(-3) - 24 = 0$ and $(8)^2 - 5(8) - 24 = 0$.

→ $x^2 - 11x + 18 = 0$:

We want to put it in the form of $(x + e)(x + f) = 0$, where:

- $e + f = -11$
- $e \times f = 18$

Let's look at the number

- $e \times f > 0$, so either
 - * e and f are positive
 - * e and f are negative
- we have $e + f = -11 < 0$, so one number at least must be negative.

It means they both must be negative.

There are three pairs of numbers that multiply together to give 18 and that are negative:

1. -1 and -18
2. -2 and -9
3. -3 and -6

Because we want them to sum up to -11 :

1. $-1 + -18 \neq -11$
2. $-2 + -9 = -11$ we can stop

It means that

$$x^2 - 11x + 18 = (x - 2)(x - 9)$$

We want to find the roots of the equation: $(x - 2)(x - 9) = 0$ means that either

- $x - 2 = 0 \Rightarrow x = 2$
- $x - 9 = 0 \Rightarrow x = 9$

You can verify that $(2)^2 - 11(2) + 18 = 0$ and $(9)^2 - 11(9) + 18 = 0$.

→ $3x^2 + 27x = 0$:

This one is slightly easier because there is not a constant term and we can factorise $3x$ out of each term.

$$3x^2 + 27x = 0 \Leftrightarrow 3x(x + 9) = 0$$

$3x$ is equivalent to $3(x - 0)$. We hence want to find the roots of the equation: $3(x - 0)(x + 9) = 0$ means that either

- $x - 0 = 0 \Rightarrow x = 0$
- $x + 9 = 0 \Rightarrow x = -9$

→ $x^2 + 7x + 14 = 0$:

We want to put it in the form of $(x + e)(x + f) = 0$, where:

- $e + f = 7$
- $e \times f = 14$

There are four pairs of numbers that multiply together to give 14:

1. 1 and 14
2. -1 and -14
3. 2 and 7

4. -2 and -7

Because we want them to sum up to 7:

1. $1 + 14 \neq 7$
2. $-1 + -14 \neq 7$
3. $2 + 7 \neq 7$
4. $-2 + -7 \neq 7$

None of the pairs add up to give 7, so this quadratic equation cannot be solved by factorization !
This example shows that not all quadratics factorise with this method.

Exercise 2.

2.1 $x^2 + 7x + 10 = 0$

2.2 $x^2 + 6x + 8 = 0$

2.3 $x^2 + 7x + 12 = 0$

2.4 $x^2 + 9x + 20 = 0$

2.5 $x^2 + 3x - 10 = 0$

2.6 $x^2 + 2x - 8 = 0$

2.7 $x^2 - 2x - 15 = 0$

2.8 $x^2 + 4x - 12 = 0$

2.9 $x^2 - 4x + 3 = 0$

2.10 $x^2 + 9x + 18 = 0$

2.11 $x^2 - 16 = 0$

2.12 $x^2 - 6x + 9 = 0$

2.13 $x^2 - 25 = 0$

2.14 $3x^2 - 48 = 0$

2.15 $8x^2 - 24x = 0$

2.16 $9x^2 - x = 0$

2.17 $x^2 + 10x - 24 = 0$

2.18 $x^2 - 10x + 16 = 0$

2.19 $x^2 - 7x - 44 = 0$

2.20 $x^2 - 8x + 12 = 0$

IV Solving quadratic equations using the formula

If a quadratic equation will not factorise (or it will, but you cannot spot that it will) then you can always use the celebrated quadratic formula.

QUADRATIC FORMULA: If $ax^2 + bx + c = 0$ then x is given by the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can prove it by the method known as *complete the square*. It is also a method to solve a quadratic equations !

The main idea is to transform $ax^2 + bx + c = 0$ in an equation of the form $(x + m)^2 = n^2$.

There is a "square" on each side of the equation, hence the name.

Let's start with $(x + m)^2 = n^2$ to go back to $ax^2 + bx + c = 0$.

$$\begin{aligned} (x + m)^2 = n^2 &\Leftrightarrow x^2 + 2mx + m^2 = n^2 && \text{removing the brackets} \\ &\Leftrightarrow x^2 + 2mx + m^2 - n^2 = 0 && \text{changing } n^2 \text{ to the left hand side} \\ &\Leftrightarrow ax^2 + \underbrace{2amx}_b + \underbrace{am^2 - an^2}_c = 0 && \text{multiplying by } a \end{aligned}$$

It means

$$\rightarrow 2am = b \text{ and, consequently, } m = \frac{b}{2a}$$

$$\rightarrow am^2 - an^2 = c \text{ and, consequently, } an^2 = am^2 - c.$$

Replacing m with its value and dividing by a leads to

$$n^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Now, reorganizing the terms under one fraction:

$$\begin{aligned} n^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\ &= \frac{b^2}{4a^2} - \frac{c}{a} \\ &= \frac{b^2 - 4ac}{4a^2} \end{aligned}$$

and hence $n = \frac{\sqrt{b^2 - 4ac}}{2a}$. We now have

$$ax^2 + bx + c = 0 \Leftrightarrow (x + m)^2 = n^2$$

It means that, taking the square root of each side:

$$x + m = \pm n \Leftrightarrow x = -m \pm n$$

Replacing m and n with their values:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Et voilà !

Not only we have shown the formula, but we also have an interesting form:

$$ax^2 + bx + c = 0 \Leftrightarrow (x + m)^2 = n^2$$

It will make sense in next session, but we actually have the *vertex form* of the parabola. If we want to plot a parabola, we have:

$$y = ax^2 + bx + c$$

or, using our expression, it is:

$$y = a(x + m)^2 - an^2 \Leftrightarrow y = a(x - h)^2 + k$$

This means that:

- $x = h$ is the extrema (either a minimum, if $a > 0$, or a maximum, if $a < 0$). The extrema coordinates is (h, k) .
- The focal point of the parabola is $\left(h, k + \frac{1}{4a}\right)$

It is really important for telescopes, satellites and for having good TV signal !

let's solve the following quadratic equations:

→ $2x^2 - 9x + 2 = 0$:

We have $a = 2, b = -9$ and $c = 2$. The formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{the formula} \\ &= \frac{-(-9) \pm \sqrt{(-9)^2 - 4 \times 2 \times 2}}{2 \times 2} && \text{subs. the values} \\ &= \frac{9 \pm \sqrt{81 - 16}}{4} && \text{simplifying} \\ &= \frac{9 \pm \sqrt{65}}{4} && \text{simplifying} \end{aligned}$$

And finally

$$x = \frac{9 - \sqrt{65}}{4} \text{ or } \frac{9 + \sqrt{65}}{4}$$

to 4 decimal places,

$$x = 0.2344 \text{ or } 4.2656$$

→

→ $3x^2 + 4x - 8 = 0$:

We have $a = 3, b = 4$ and $c = -8$. The formula gives:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{the formula} \\ &= \frac{-4 \pm \sqrt{4^2 - 4 \times 3 \times (-8)}}{2 \times 3} && \text{subs. the values} \\ &= \frac{-4 \pm \sqrt{16 + 96}}{6} && \text{simplifying} \\ &= \frac{-4 \pm \sqrt{112}}{6} && \text{simplifying} \end{aligned}$$

And finally

$$x = \frac{-4 - \sqrt{112}}{6} \text{ or } \frac{-4 + \sqrt{112}}{6}$$

to 4 decimal places,

$$x = -2.4305 \text{ or } 1.0972$$

Exercise 3.

Solve the following by the formula giving your answers to 2 decimal places where appropriate:

3.1 $x^2 - 4x - 3 = 0$

3.2 $3x^2 + 5x + 1 = 0$

3.3 $x^2 - 9 = 0$

3.4 $2x^2 - 14x + 4 = 0$

3.5 $x^2 + 4x - 8 = 0$

3.6 $5x^2 + 4x - 3 = 0$

3.7 $4x^2 - 7x - 1 = 0$

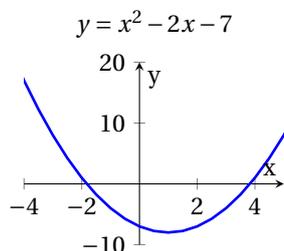
3.8 $x^2 - 6x - 3 = 0$

3.9 $6x^2 + 8x + 2 = 0$

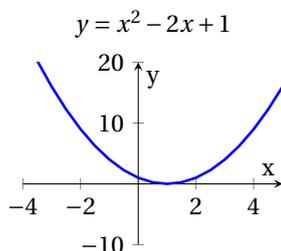
3.10 $3x^2 - 4x - 6 = 0$

V Solutions to exercises

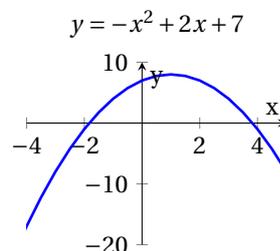
Solution 1.



(1)



(2)



(3)

1.1 $y = x^2 - 2x - 7: x_1 \approx -1.8, x_2 \approx 3.8$

1.2 $y = x^2 - 2x + 1: x = 1$

1.3 $y = -x^2 + 2x + 7: x_1 \approx -1.8, x_2 \approx 3.8$

Solution 2.

2.1 $x = -2$ or $x = -5$

2.2 $x = -2$ or $x = -4$

2.3 $x = -3$ or $x = -4$

2.4 $x = -4$ or $x = -5$

2.5 $x = 2$ or $x = -5$

2.6 $x = 2$ or $x = -4$

2.7 $x = 5$ or $x = -3$

2.8 $x = 2$ or $x = -6$

2.9 $x = 3$ or $x = 1$

2.10 not possible by factorization

2.11 $x = 4$ or $x = -4$

2.12 $x = 3$ or $x = 3$

2.13 $x = 5$ or $x = -5$

2.14 $x = 4$ or $x = -4$

2.15 $x = 3$ or $x = 0$

2.16 $x = \frac{1}{9}$ or $x = 0$

2.17 $x = 2$ or $x = -12$

2.18 $x = 8$ or $x = 2$

2.19 $x = 11$ or $x = -4$

2.20 $x = 6$ or $x = 2$

Solution 3.

3.1 $x = 4.65$ or $x = -0.65$

3.2 $x = -0.23$ or $x = -1.43$

3.3 $x = 3$ or $x = -3$

3.4 $x = 6.70$ or $x = 0.30$

3.5 $x = 1.46$ or $x = -5.46$

3.6 $x = 0.47$ or $x = -1.27$

3.7 $x = 1.88$ or $x = -0.13$

3.8 $x = 6.46$ or $x = -0.46$

3.9 $x = -0.33$ or $x = -1$

3.10 $x = 2.23$ or $x = -0.90$

Bibliography

[Croft and Davidson, 2016] Croft, A. and Davidson, R. (2016). *Foundation Maths*. Pearson.